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Quantum duality under the θ-exact Seiberg-Witten map

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Abstract: We show that in the perturbative regime defined by the coupling constant, the θ-exact Seiberg-Witten map applied to the noncommutative U(N) Yang-Mills — with or without Supersymmetry — gives an ordinary gauge theory which is, at the quantum level, dual to the former. We do so by using the on-shell DeWitt effective action and dimensional regularization. We explicitly compute the one-loop two-point function contribution to the on-shell DeWitt effective action of the ordinary U(1) theory furnished by the θ-exact Seiberg-Witten map. We find that the non-local UV divergences found in the propagator in the Feynman gauge all but disappear, so that they are not physically relevant. We also show that the quadratic noncommutative IR divergences are gauge-fixing independent and go away in the Supersymmetric version of the U(1) theory.

Keywords: BRST Quantization, Non-Commutative Geometry, Supersymmetric gauge theory

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1 Introduction

It was found in [1] that noncommutative U(N) Yang-Mills theories admit two dual formulations at the classical level: one, in terms of noncommutative gauge fields, the other by using ordinary gauge fields. One moves between these two dual descriptions of the same theory by employing the so-called Seiberg-Witten (SW) map. This map maps ordinary U(N) gauge fields into noncommutative U(N) gauge fields and viceversa [2, 3]. Whether this duality persists at the quantum level is still an open issue, even at the one-loop level. Indeed, a key feature of noncommutative field theories defined by using noncommutative fields is the UV/IR mixing phenomenon unveiled in [4–7]; a phenomenon which cannot be seen by defining the action of the would-be dual ordinary theory as the expansion in the noncommutativity matrix, $\theta^{\mu\nu}$, furnished by the Seiberg-Witten map, with the latter constructed as a formal power series in $\theta^{\mu\nu}$. It was not known how to reproduce the UV/IR mixing effect in the formulation of noncommutative gauge theory in terms of ordinary fields until the paper in [8] was issued. In accord with the very essence of the perturbative coupling constant description of the quantum field theory our approach to the Seiberg-Witten map issue, is to built the SW map by using the expansion in terms of the coupling constant [8, 9] (and no expansion in $\theta^{\mu\nu}$ is carried out). Thus, the $\theta^{\mu\nu}$ dependence of the perturbative, in the coupling constant, definition of the theory is treated in an exact way and, then, the UV/IR mixing effect pops up. The occurrence of the UV/IR mixing phenomenon in both these quantum field theories gives strong support to the idea that they are dual descriptions of the same underlying quantum field theory, at least in the perturbative regime defined by the coupling constant. And yet, in U(1) Yang-Mills theory, the UV divergent part of the two-point function of the noncommutative gauge field is local, whereas the UV divergent bit of the two-point function of the ordinary theory obtained by using the θ-exact Seiberg-Witten map contains unusual θ-dependent nonlocal contributions, at least in the Feynman gauge. These nonlocal contributions where unearthed in [10–12], and their existence casts doubts on the truth of the quantum duality conjecture at hand. Of course, UV divergent contributions to the two-point function are, in general, gauge dependent; so to decide whether the duality conjecture is right or wrong, there remains to be seen whether or not those nonlocal terms are really gauge dependent; since the gauge dependent contributions are not physically relevant.

It is known that in the θ-unexpanded noncommutative nonsupersymmetric gauge theory defined in terms of the noncommutative fields, the noncommutative quadratic IR divergence induced by UV/IR mixing signals an IR instability [13]; and that this IR instability can be cured by making the theory supersymmetric, since supersymmetry removes the corresponding quadratic noncommutative IR divergences [14, 15]. Furthermore, it was shown in [16] that if the noncommutative fields carry a linear realization of Supersymmetry their ordinary duals under the Seiberg-Witten map carry a nonlinear realization of Supersymmetry. Hence, it is far from trivial that the Supersymmetry cancelation mechanism between the one-loop noncommutative quadratic IR divergences coming from bosonic and fermionic degrees of freedom works when the classical noncommutative theory is formulated, first, in terms of the ordinary fields and then quantized. And yet, it has been shown in [17] that the

Supersymetry cancelation mechanism just mentioned works for all the two-point functions when we have $\mathcal{N} = 1, 2$ and 4 Supersymmetry. This result gives further robustness to the quantum duality conjecture between the formulation in terms of ordinary fields and the description in terms of noncommutative fields. However, the nonlocal UV divergent structure still persists after introducing Supersymmetry in the game. But, by using two different gauge-fixing terms, it was shown in [17] that the nonlocal UV divergent contributions are gauge dependent and, therefore, it could be possible to remove them. This is unlike the noncommutative quadratic IR divergences which do not change with the gauge-fixing term as proved in [15], in the noncommutative field description, and in [17], in the ordinary field formulation, respectively.

Now, since it is known that the on-shell DeWitt effective action is independent of the gauge-fixing term used to define the path integral, thus this action can be used to compute the S matrix elements. Hence, by using on-shell DeWitt effective action [18–21], one could settle the question of the physical relevance of the nonlocal UV divergent terms found in the two-point functions of the noncommutative theory formulated in terms of the ordinary fields, and as a bonus obtain a complete proof of the gauge-fixing independence of the UV/IR mixing phenomenon and also the cancelation of the noncommutative quadratic IR divergences achieved by introducing Supersymmetry.

Let $S_{\rm NCYM}[\hat{A}_\mu]$ denote the classical action of noncommutative U(N) Yang-Mills theory, where $\hat{A}_\mu$ is a noncommutative gauge field configuration. Let $\hat{\Gamma}_{\rm DeW}[\hat{B}_\mu]$ stand for the on-shell DeWitt effective action of noncommutative field theory whose action is $S_{\rm NCYM}[\hat{A}_\mu]$. Let $\Gamma_{\rm DeW}[B_\mu]$ be the symbol for the on-shell DeWitt effective action of the ordinary — i.e., defined in terms of ordinary fields — U(N) gauge theory whose classical action is $S_{\rm NCYM}\Big[\hat{A}_\mu[A_\mu]\Big]$, $\hat{A}_\mu[A_\mu]$ being the Seiberg-Witten map that relates the ordinary U(N) gauge field A_μ with the noncommutative U(N) gauge field $\hat{A}_\mu$. Then, the purpose of this paper is to show that, at any loop order and in dimensional regularization,

$$\hat{\Gamma}_{\rm DeW}\Big[\hat{B}_\mu[B_\mu]\Big] = \Gamma_{\rm DeW}[B_\mu], \tag{1.1}$$

where $\hat{B}_\mu[B_\mu]$ stands for the Seiberg-Witten map between the ordinary U(N) gauge field B_μ with the noncommutative U(N) gauge field $\hat{B}_\mu$. Thus proving that the conjecture that Seiberg-Witten map provides two dual formulations of the same underlying quantum theory is true for the noncommutative U(N) Yang-Mills theories, with or without Supersymmetry.

Let us warn the reader that to avoid any clashes with unitarity [22], we shall always consider $\theta^{\mu\nu}$ — the noncommutativity matrix — such that $\theta^{0i} = 0$, $i = 1, 2, 3$.

The layout of this paper is as follows. In section 2, we introduce the on-shell DeWitt action for noncommutative U(N) Yang-Mill theory and the corresponding ordinary U(N) gauge theory defined by means of the θ-exact Seiberg-Witten map. In the same section, we also discuss the effect and some properties of the Seiberg-Witten map applied to the ordinary background-field splitting. We establish the quantum equivalence of the field theories defined in the previous section by performing the appropriate changes of variables in the path integral in section 3, while in section 4, we check the conclusion reached in section 3 by direct computation — i.e., by using the Feynman rules (FR's) derived from classical action — of the one-loop two-point function of the on-shell DeWitt action of dual ordinary U(1) gauge theory: we show by explicit computation that, in particular, all the ugly non-local UV divergences — see [17] — that occur in the propagator in the Feynman gauge all but go away. The noncommutative quadratic IR divergences remain unless the noncommutative theory is supersymmetric with $\mathcal{N}=1,2$ and 4 Supersymmetry. Of course, there is no UV divergence nor any noncommutative IR divergence (quadratic or logarithmic) for $\mathcal{N}=4$ U(1) super Yang-Mills. We have included appendices to help understanding the central body of the paper.

2 Seiberg-Witten map and DeWitt effective action

In this and next section we provide an expanded proof/review of the equivalence between DeWitt effective actions in terms of noncommutative and ordinary fields via θ-exact Seiberg-Witten map indicated in [23].

2.1 DeWitt action and the path integral in terms of noncommutative fields

Let $\hat{A}_\mu = \hat{B}_\mu + \hbar^{\frac{1}{2}}\hat{Q}_\mu$ be the standard splitting of the noncommutative U(N) gauge field $\hat{A}_\mu$ in a noncommutative background $\hat{B}_\mu$ and a noncommutative quantum field $\hat{Q}_\mu$. We shall assume that $\hat{B}_\mu$ satisfies the classical noncommutative equations of motion (EOM), which read $\hat{D}_\mu\hat{F}^{\mu\nu}=0$. Then the on-shell DeWitt effective action [21], $\hat{\Gamma}_{\mathrm{DeW}}\big[\hat{B}_\mu\big]$, is given by the following path integral

$$e^{\frac{i}{\hbar}\hat{\Gamma}_{\mathrm{DeW}}\left[\hat{B}_\mu\right]} = \int d\hat{Q}^a_\mu d\hat{C}^a d\hat{\bar{C}}^a d\hat{F}^a \; e^{\frac{i}{\hbar}S_{\mathrm{NCYM}}\left[\hat{B}_\mu+\hbar^{\frac{1}{2}}\hat{Q}_\mu\right]+iS_{\mathrm{BFG}}\left[\hat{B}_\mu,\hat{Q}_\mu,\hat{F},\hat{\bar{C}},\hat{C}\right]}. \tag{2.1}$$

The gauge-fixing term $S_{\mathrm{BFG}}\big[\hat{B}_\mu,\hat{Q}_\mu,\hat{F},\hat{\bar{C}},\hat{C}\big]$ is[1]

$$\begin{aligned} S_{\mathrm{BFG}}\big[\hat{B}_\mu,\hat{Q}_\mu,\hat{F},\hat{\bar{C}},\hat{C}\big] &= \frac{\hbar^{\frac{1}{2}}}{g^2}\int \mathrm{tr}\,\hat{\delta}_{\mathrm{BRS}}\,\hat{\bar{C}}\left(\alpha\hat{F}+\hat{D}_\mu[\hat{B}_\mu]\hat{Q}^\mu\right) \\ &= \frac{1}{g^2}\int \mathrm{tr}\,\left(\alpha\hat{F}\star\hat{F}-\hat{\bar{C}}\hat{D}_\mu[\hat{B}_\mu]\hat{D}^\mu[\hat{B}_\mu+\hbar^{\frac{1}{2}}\hat{Q}_\mu]\hat{C}\right) \end{aligned} \tag{2.2}$$

where $\hat{\delta}_{\mathrm{BRS}}$ stands for the noncommutative BRS operator, which acts on the noncommutative fields as follows:

$$\begin{aligned} &\hat{\delta}_{\mathrm{BRS}}\hat{B}_\mu = 0, \qquad \hat{\delta}_{\mathrm{BRS}}\hat{Q}_\mu = \hbar^{-\frac{1}{2}}\hat{D}_\mu[\hat{B}_\mu+\hbar^{\frac{1}{2}}\hat{Q}_\mu]\hat{C}, \\ &\hat{\delta}_{\mathrm{BRS}}\hat{C} = -i\hat{C}\star\hat{C}, \quad \hat{\delta}_{\mathrm{BRS}}\hat{\bar{C}} = \hbar^{-\frac{1}{2}}\hat{F}, \quad \hat{\delta}_{\mathrm{BRS}}\hat{F} = 0. \end{aligned} \tag{2.3}$$

Although — as shown in [24] by using BRS techniques for ordinary theories, a proof which remains valid in the case at hand — $\hat{\Gamma}_{\mathrm{DeW}}\big[\hat{B}_\mu\big]$ does not depend on the choice of gauge-fixing term, we have chosen the background field gauge (BFG) for convenience.

[1] As usual, we use tr to denote trace over the Lie algebra generators, while Tr for functional trace.

2.2 DeWitt effective action of the dual classical ordinary theory

The next task will be the background field quantization of the ordinary theory with action

$$S_{\rm NCYM}[A_\mu] = -\frac{1}{4g^2}\int \mathrm{tr}\left(\hat F_{\mu\nu}\left[\hat A_\mu\left[A_\mu\right]\right]\hat F^{\mu\nu}\left[\hat A_\mu\left[A_\mu\right]\right]\right),$$

where $\hat A_\mu[A_\mu]$ is the θ-exact Seiberg-Witten map which expressed the noncommutative field $\hat A_\mu$ in terms of its ordinary counterpart A_μ. But before carrying out the background field quantization, we need to discuss — since we are dealing with a nonlinear map — how the Seiberg-Witten map acts on the background-field-quantization splitting, $A_\mu = B_\mu + \hbar^{\frac12} Q_\mu$, of the ordinary gauge field; this we do next.

2.2.1 Seiberg-Witten map and the background-field splitting

Let T^a denote the generators of U(N) in the fundamental representation

$$\mathrm{tr}\, T^aT^b = \delta^{ab},\ \ \left[T^a, T^b\right] = i f^{abc}T^c. \tag{2.4}$$

Here $\hat A_\mu = \hat A^a_\mu T^a$ is the noncommutative gauge field, the $A_\mu = A^a_\mu T^a$ is the ordinary gauge field, the $\hat C = \hat C^a T^a$ is the NC ghost field and the $C = C^aT^a$ is the ordinary ghost field, all in terms of components fields. The BRS transformations of $\hat A_\mu$, $\hat C$, A_μ and C read

$$\hat\delta_{\rm BRS}\hat A_\mu = \hat D_\mu\left[\hat A_\mu\right]\hat C = \partial_\mu \hat C + i\left[\hat A_\mu \stackrel{\star}{,} \hat C\right],\quad \hat\delta_{\rm BRS}\hat C = -i\hat C\star\hat C, \tag{2.5}$$

$$\delta_{\rm BRS}A_\mu = D_\mu\left[A_\mu\right]C = \partial_\mu C + i\left[A_\mu, C\right],\quad \delta_{\rm BRS}C = -iC\cdot C. \tag{2.6}$$

The Seiberg-Witten (SW) map

$$\hat A_\mu = \hat A_\mu\left[A_\mu,\theta\right],\ \ \hat C = \hat C\left[A_\mu, C, \theta\right], \tag{2.7}$$

is a solution to the following equations

$$\hat\delta_{\rm BRS}\hat A_\mu = \delta_{\rm BRS}\hat A_\mu\left[A_\mu,\theta\right],\ \ \hat\delta_{\rm BRS}\hat C = \delta_{\rm BRS}\hat C\left[A_\mu, C,\theta\right]. \tag{2.8}$$

One can expand the Seiberg-Witten map θ-exactly -see [25, 26]:

$$\hat A_\mu\left[A_\mu,\theta\right](x) = A_\mu(x) + \sum_{n=2}^{\infty}\mathcal{A}^{(n)}_\mu(x), \tag{2.9}$$

$$\hat C\left[A_\mu, C,\theta\right](x) = C(x) + \sum_{n=1}^{\infty}\mathcal{C}^{(n)}(x), \tag{2.10}$$

where

$$\begin{aligned}\mathcal{A}^{(n)}_\mu(x) = \int\prod_{i=1}^{n}\frac{d^4p_i}{(2\pi)^4}e^{i\left(\sum\limits_{i=1}^{n}p_i\right)x}\mathfrak{A}^{(n)}_\mu\left[(a_1,\mu_1,p_1),\dots,(a_n,\mu_n,p_n);\theta\right]\\ \cdot\tilde A^{a_1}_{\mu_1}(p_1)\dots\tilde A^{a_n}_{\mu_n}(p_n),\end{aligned} \tag{2.11}$$

$$\begin{aligned}\mathcal{C}^{(n)}(x) = \int\prod_{i=1}^{n}\frac{d^4p_i}{(2\pi)^4}e^{i\left(p+\sum\limits_{i=1}^{n}p_i\right)x}\mathfrak{C}^{(n)}\left[(a_1,\mu_1,p_1),\dots,(a_n,\mu_n,p_n);(a,p);\theta\right]\\ \cdot\tilde A^{a_1}_{\mu_1}(p_1)\dots\tilde A^{a_n}_{\mu_n}(p_n)C^a(p),\end{aligned} \tag{2.12}$$

$\mathfrak{A}_{\mu}^{(n)}$ and $\mathfrak{C}^{(n)}$ are totally symmetric under the permutations with respect to the set of the parameter-triples $\{(a_i, \mu_i, p_i)|i = 1, \ldots, n\}$, which have the property — of key importance in our later discussion — that only the momenta which are not contracted with $\theta^{\mu\nu}$ build up polynomials which never occur in the denominator [25, 26].

Now, let us introduce the ordinary background-field splitting

$$A_\mu = B_\mu + \hbar^{\frac{1}{2}} Q_\mu, \tag{2.13}$$

where B_μ is the background field and Q_μ the quantum fluctuation. Substituting (2.13) into (2.9)–(2.12), one gets

$$\hat{A}_\mu\big[B_\mu + \hbar^{\frac{1}{2}} Q_\mu, \theta\big] = \hat{B}_\mu\big[B_\mu, \theta\big] + \hbar^{\frac{1}{2}} \hat{Q}_\mu\big[B_\mu, Q_\mu, \hbar, \theta\big], \tag{2.14}$$

$$\hat{C}\big[B_\mu + \hbar^{\frac{1}{2}} Q_\mu, C, \theta\big] = \hat{C}\big[B_\mu, C, \theta\big] + \hbar^{\frac{1}{2}} \hat{C}^{(1)}\big[B_\mu, Q_\mu, C, \hbar, \theta\big], \tag{2.15}$$

where

$$\hat{B}_\mu\big[B_\mu, \theta\big] = B_\mu + \sum_{n=2}^{\infty} \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} e^{i\left(\sum_{i=1}^{n} p_i\right)x} \mathfrak{A}_{\mu}^{(n)}\big[(a_1, \mu_1, p_1), \ldots, (a_n, \mu_n, p_n); \theta\big] \cdot \tilde{B}_{\mu_1}^{a_1}(p_1) \ldots \tilde{B}_{\mu_n}^{a_n}(p_n), \tag{2.16}$$

$$\hat{Q}_\mu\big[B_\mu, Q_\mu, \hbar, \theta\big] = Q_\mu + \sum_{n=2}^{\infty} \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} e^{i\left(\sum_{i=1}^{n} p_i\right)x} \mathfrak{A}_{\mu}^{(n)}\big[(a_1, \mu_1, p_1), \ldots, (a_n, \mu_n, p_n); \theta\big] \cdot \left(\sum_{m=1}^{n} \hbar^{\frac{m-1}{2}} \frac{n!}{m!(n-m)!} \tilde{Q}_{\mu_1}^{a_1}(p_1) \ldots \tilde{Q}_{\mu_m}^{a_m}(p_m) \tilde{B}_{\mu_{m+1}}^{a_m+1}(p_{m+1}) \ldots \tilde{B}_{\mu_n}^{a_n}(p_n)\right), \tag{2.17}$$

$$\hat{C}\big[B_\mu, C, \theta\big] = C(x) + \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} e^{i\left(p + \sum_{i=1}^{n} p_i\right)x} \cdot \mathfrak{C}^{(n)}\big[(a_1, \mu_1, p_1), \ldots, (a_n, \mu_n, p_n); (a, p); \theta\big] \tilde{B}_{\mu_1}^{a_1}(p_1) \ldots \tilde{B}_{\mu_n}^{a_n}(p_n) C^a(p)\,, \tag{2.18}$$

$$\hat{C}^{(1)}\big[B_\mu, Q_\mu, C, \hbar, \theta\big] = \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} e^{i\left(p + \sum_{i=1}^{n} p_i\right)x} \cdot \mathfrak{C}^{(n)}\big[(a_1, \mu_1, p_1), \ldots, (a_n, \mu_n, p_n); (a, p); \theta\big] \cdot \left(\sum_{m=1}^{n} \hbar^{\frac{m-1}{2}} \frac{n!}{m!(n-m)!} \tilde{Q}_{\mu_1}^{a_1}(p_1) \ldots \tilde{Q}_{\mu_m}^{a_m}(p_m) \tilde{B}_{\mu_{m+1}}^{a_{m+1}}(p_{m+1}) \ldots \tilde{B}_{\mu_n}^{a_n}(p_n) C^a(p)\right). \tag{2.19}$$

In the previous equations the convention $\tilde{B}_{\mu_{n+1}}^{a_{n+1}}(p_{n+1}) = 1$ is assumed.

Let us stress that $\hat{B}_\mu\big[B_\mu, \theta\big]$ and $\hat{C}\big[B_\mu, C, \theta\big]$ are standard Seiberg-Witten maps, i.e., are solutions to the equations in (2.8), when in the latter A_μ has been replaced with B_μ. However, $Q_\mu\big[B_\mu, Q_\mu, \hbar, \theta\big]$ and $C^{(1)}\big[B_\mu, Q_\mu, C, \hbar, \theta\big]$ are not standard Seiberg-Witten maps, for they are solutions to

$$\begin{aligned} \hat{\delta}_{\rm BRS} \hat{Q}_\mu &= \delta_{\rm BRS} \hat{Q}_\mu\big[B_\mu, Q_\mu, \hbar, \theta\big], \\ \hat{\delta}_{\rm BRS} \hat{C} &= \delta_{\rm BRS} \hat{C}\big[B_\mu + \hbar^{\frac{1}{2}} Q_\mu, C, \theta\big], \end{aligned} \tag{2.20}$$

where the splitting of $\hat{C}\big[B_\mu + \hbar^{\frac{1}{2}}Q_\mu, C, \theta\big]$ is given/defined in (2.15), and

$$\begin{aligned}
\delta_{\rm BRS} B_\mu &= 0\,,\\
\delta_{\rm BRS} Q_\mu &= \hbar^{-\frac{1}{2}} D_\mu\big[B_\mu + \hbar^{\frac{1}{2}} Q_\mu\big] C\,,\\
\delta_{\rm BRS} C &= -iC\cdot C\,,\\
\hat{\delta}_{\rm BRS}\hat{Q}_\mu &= \hbar^{-\frac{1}{2}} \hat{D}_\mu\big[\hat{B}_\mu + \hbar^{\frac{1}{2}}\hat{Q}_\mu\big]\hat{C}\,,\\
\hat{\delta}_{\rm BRS}\hat{C}\big[B_\mu + \hbar^{\frac{1}{2}}Q_\mu, C, \theta\big] &= -i\hat{C}\star\hat{C}\,.
\end{aligned} \tag{2.21}$$

In view of equation (2.20), $\hat{Q}_\mu\,[B_\mu, Q_\mu, \hbar, \theta]$ can be called the Seiberg-Witten map of the quantum field $\hat{Q}_\mu$ in the presence of the background field $\hat{B}_\mu$.

That $\hat{Q}_\mu\big[B_\mu, Q_\mu, \hbar, \theta\big]$ satisfies (2.20) is a consequence of the fact that it is defined in terms of $\hat{A}_\mu\big[A_\mu, \theta\big]$ as done in (2.14) and that $\hat{A}_\mu\big[A_\mu, \theta\big]$, along with $\hat{C}\big[A_\mu, C, \theta\big]$, solves the Seiberg-Witten equations in (2.8). Indeed,

$$\begin{aligned}
\hbar^{\frac{1}{2}}\hat{Q}_\mu\big[B_\mu, Q_\mu, \hbar, \theta\big] &= \hat{A}_\mu\big[B_\mu + \hbar^{\frac{1}{2}}Q_\mu, \theta\big] - \hat{B}_\mu\big[B_\mu, \theta\big] \;\;\Rightarrow\\
\hbar^{\frac{1}{2}}\delta_{\rm BRS}\hat{Q}_\mu\big[B_\mu, Q_\mu, \hbar, \theta\big] &= \delta_{\rm BRS}\hat{A}_\mu\big[B_\mu + \hbar^{\frac{1}{2}}Q_\mu, \theta\big] - \delta_{\rm BRS}\hat{B}_\mu\big[B_\mu, \theta\big]\\
&= \delta_{\rm BRS}\hat{A}_\mu\big[B_\mu + \hbar^{\frac{1}{2}}Q_\mu, \theta\big]\\
&= \hat{D}_\mu\Big[\hat{A}_\mu\big[B_\mu + \hbar^{\frac{1}{2}}Q_\mu, \theta\big]\Big]\hat{C}\big[B_\mu + \hbar^{\frac{1}{2}}Q_\mu, C, \theta\big].
\end{aligned}$$

Using the results displayed above, one can show that

$$\delta^2_{\rm BRS}\hat{Q}_\mu\big[B_\mu, Q_\mu, \hbar, \theta\big] = \hat{\delta}^2_{\rm BRS}\hat{Q}_\mu = 0, \tag{2.22}$$

$$\delta^2_{\rm BRS}\hat{C}\big[B_\mu + \hbar^{\frac{1}{2}}Q_\mu, C, \theta\big] = \hat{\delta}^2_{\rm BRS}\hat{C} = 0. \tag{2.23}$$

Before closing this subsection, for later use we shall show that both above terms, $\mathfrak{A}^{(n)}_\mu\big[(a_1, \mu_1, p_1), \ldots, (a_n, \mu_n, p_n); \theta\big]$ and $\mathfrak{C}^{(n)}\big[(a_1, \mu_1, p_1), \ldots, (a_n, \mu_n, p_n); (a, p); \theta\big]$ defining all Seiberg-Witten maps introduced in this section, are linear combinations of functions of the type

$$\mathbb{Q}(p_1, \ldots, p_n)\cdot\mathbb{K}(p_i\theta p_j), \tag{2.24}$$

where $\mathbb{Q}(p_1, \ldots, p_n)$ is a monomial of the momenta p_i and $\mathbb{K}(p_i\theta p_j)$ is a function of the variables $p_i\theta p_j$, $i, j = 1 \ldots n$, only. We use well known notation $q\theta k = q_\mu\theta^{\mu\nu}k_\nu$.

Let us begin with $\mathfrak{A}^{(n)}_\mu\big[(a_1, \mu_1, p_1), \ldots, (a_n, \mu_n, p_n); \theta\big]$. In [25], it was shown that the θ-exact Seiberg-Witten map can be constructed by setting $h = 1$ in the following formal series

$$A_\mu[a_\rho; h\theta] = \sum_{n=1}^{\infty}\mathcal{A}^{(n)}_\mu[a_\rho; h\theta], \tag{2.25}$$

where $\mathcal{A}^{(n)}_\mu[a_\rho; h\theta]$ is of order n in the number of classical fields and it is given by the recursive solution to the following set of equations

$$\begin{aligned}
\mathcal{A}_\mu^{(1)}[a_\rho;h\theta] &= a_\mu, \forall h,\\
\mathcal{A}_\mu^{(2)}[a_\rho;h\theta] &= \int_0^h dt\,\Big(\frac{1}{2}\,\theta^{ij}\{\mathcal{A}_i^{(1)},\partial_j\mathcal{A}_\mu^{(1)}\}_{\star_t}-\frac{1}{4}\,\theta^{ij}\{\mathcal{A}_i^{(1)},\partial_\mu\mathcal{A}_j^{(1)}\}_{\star_t}\Big),\\
\mathcal{A}_\mu^{(3)}[a_\rho;h\theta] &= \int_0^h dt\,\Big(\frac{1}{2}\,\theta^{ij}\{\mathcal{A}_i^{(1)},\partial_j\mathcal{A}_\mu^{(2)}[a_\rho;t\theta]\}_{\star_t}+\frac{1}{2}\,\theta^{ij}\{\mathcal{A}_i^{(2)}[a_\rho;t\theta],\partial_j\mathcal{A}_\mu^{(1)}\}_{\star_t}\\
&\qquad -\frac{1}{4}\,\theta^{ij}\{\mathcal{A}_i^{(2)}[a_\rho;t\theta],\partial_\mu\mathcal{A}_j^{(1)}\}_{\star_t}-\frac{1}{4}\,\theta^{ij}\{\mathcal{A}_i^{(1)},\partial_\mu\mathcal{A}_j^{(2)}[a_\rho;t\theta]\}_{\star_t}\\
&\qquad +\frac{i}{4}\,\theta^{ij}\{\mathcal{A}_i^{(1)},[\mathcal{A}_j^{(1)},\mathcal{A}_\mu^{(1)}]_{\star_t}\}_{\star_t},\Big),\\
&\ \ \cdots\\
\mathcal{A}_\mu^{(n)}[a_\rho;h\theta] &= \int_0^h dt\,\Big(\frac{1}{2}\,\theta^{ij}\sum_{m_1+m_2=n}\{\mathcal{A}_i^{(m_1)},\partial_j\mathcal{A}_\mu^{(m_2)}\}_{\star_t}\\
&\qquad -\frac{1}{4}\,\theta^{ij}\sum_{m_1+m_2=n}\{\mathcal{A}_i^{(m_1)},\partial_\mu\mathcal{A}_j^{(m_2)}\}_{\star_t}\\
&\qquad +\frac{i}{4}\,\theta^{ij}\sum_{m_1+m_2+m_3=n}\{\mathcal{A}_i^{(m_1)},[\mathcal{A}_j^{(m_2)},\mathcal{A}_\mu^{(m_3)}]_{\star_t}\}_{\star_t}\Big),\ \forall n>3.
\end{aligned} \tag{2.26}$$

Here $\mathcal{A}_\mu^{(m)}$ is a shorthand for $\mathcal{A}_\mu^{(m)}[a_\rho;h\theta]$.

Now, it is easily seen by inspection of the formulae given in [25] that, indeed, $\mathcal{A}_\mu^{(1)}[a_\rho;h\theta]$ and $\mathcal{A}_\mu^{(2)}[a_\rho;h\theta]$ are, after setting h=1, the Fourier transforms of a linear combination of functions of the type displayed in (2.24) multiplied by one or two ordinary gauge fields, respectively. Further, in $\mathcal{A}_\mu^{(1)}[a_\rho;h\theta]$ and $\mathcal{A}_\mu^{(2)}[a_\rho;h\theta]$, h only occurs in exponentials of the type

$$e^{\pm i\frac{h}{2}\sum\limits_{(i_1,i_2)}p_{i_1}\theta\, p_{i_2}}. \tag{2.27}$$

Notice that in $\mathcal{A}_\mu^{(1)}[a_\rho;h\theta]$ and $\mathcal{A}_\mu^{(2)}[a_\rho;h\theta]$ there is no polynomial dependence in h, but, we shall allow for the possibility that for higher n there is a cancelation among phase factors that gives rise upon integration over t to positive powers of h. Before we go on, let us recall that, for all integers $s\geq 0$, we have

$$\int_0^h dt\; t^s\,e^{At}=\frac{e^{Ah}}{A}\sum_{k=0}^{s}(-1)^{2s-k}\frac{s!}{(s-k)!A^k}h^{s-k}-(-1)^s\frac{s!}{A^{s+1}}. \tag{2.28}$$

Next, let us assume that, for all $m<n$ we have that, *a)* $\mathcal{A}_\mu^{(m)}[a_\rho;h\theta]$ is, for $h=1$, the Fourier transform of a linear combination of functions of the type displayed in (2.24) multiplied by m ordinary gauge fields and that, *b)* the h-dependence in $\mathcal{A}_\mu^{(m)}[a_\rho;h\theta]$ only occurs through functions of the form

$$h^\alpha\, e^{\pm i\frac{h}{2}\sum\limits_{(i_1,i_2)}p_{i_1}\theta\, p_{i_2}}\quad \text{or}\quad h^\beta, \tag{2.29}$$

with $\alpha \geq 0$ and $\beta \geq 0$ being integers. Then, last equation in (2.26) tell us that *a*) and *b*) hold for $\mathcal{A}_\mu^{(n)}[a_\rho; h\theta]$, so that mathematical induction leads to the conclusion that *a*) and *b*) also hold for any n; which in turn implies that $\mathfrak{A}_\mu^{(n)}\big[(a_1,\mu_1,p_1),\dots,(a_n,\mu_n,p_n);\theta\big]$ is a linear combination of functions of the type (2.24), for whatever value of n.

It is plain that the same kind of reasoning can be carried out to show that

$$\mathfrak{C}^{(n)}\big[(a_1,\mu_1,p_1),\dots,(a_n,\mu_n,p_n);(a,p);\theta\big], \tag{2.30}$$

is also a linear combination of functions of the type (2.24).

2.2.2 DeWitt effective action for ordinary fields

We are now ready to quantize the classical ordinary U(N) gauge theory which is dual, under the θ-exact Seiberg-Witten map, to the noncommutative U(N) Yang-Mills theory. To quantize the ordinary theory in question, we shall use the background-field splitting; so the classical action that defines the ordinary theory reads

$$S_{\rm NCYM}\big[B_\mu + \hbar^{\frac{1}{2}} Q_\mu\big] = -\frac{1}{4g^2}\int \operatorname{tr}\Big(\hat{F}_{\mu\nu}\big[\hat{B}_\mu + \hbar^{\frac{1}{2}}\hat{Q}_\mu\big]\hat{F}^{\mu\nu}\big[\hat{B}_\mu + \hbar^{\frac{1}{2}}\hat{Q}_\mu\big]\Big), \tag{2.31}$$

with $\hat{B}_\mu = \hat{B}_\mu\big[B_\mu\big]$ and $\hat{Q}_\mu = \hat{Q}_\mu\big[B_\mu, Q_\mu, \hbar, \theta\big]$ are the Seiberg-Witten map — standard and in the presence of a background — introduced in section 2.2.1.

Let us first introduce two extra fields, $\hat{F} = \hat{F}^a T^a$, $\hat{\bar{C}} = \hat{\bar{C}}^a T^a$, on which the ordinary $\delta_{\rm BRS}$, and noncommutative $\hat{\delta}_{\rm BRS}$, operators act by the following definition:

$$\delta_{\rm BRS}\hat{\bar{C}} = \hat{\delta}_{\rm BRS}\hat{\bar{C}} = \hbar^{-\frac{1}{2}}\hat{F}, \delta_{\rm BRS}\hat{F} = \hat{\delta}_{\rm BRS}\hat{F} = 0, \delta^2_{\rm BRS}\hat{\bar{C}} = \hat{\delta}^2_{\rm BRS}\bar{C} = \delta^2_{\rm BRS}\hat{F} = \hat{\delta}^2_{\rm BRS}\hat{F} = 0. \tag{2.32}$$

Here $\hat{\bar{C}}$ is a Grassmann field and $\hat{F}$ is a boson field. Recall that T^a is U(N) generator in the fundamental representation.

We shall assume from now on that B_μ is a solution to the classical equation of motion of the theory with action $S_{\rm NCYM}\left[B_\mu\right]$, as defined previously. Then, as shown in the appendix A, B_μ satisfies:

$$\hat{D}_\mu\big[\hat{B}_\mu[B_\mu]\big]\hat{F}^{\mu\nu}\big[\hat{B}_\mu[B_\mu]\big] = 0. \tag{2.33}$$

The on-shell DeWitt action, $\Gamma_{\rm DeW}\left[B_\mu\right]$, of ordinary theory now reads

$$e^{\frac{i}{\hbar}\Gamma_{\rm DeW}\left[B_\mu\right]} = \int dQ^a_\mu dC^a d\hat{\bar{C}}^a d\hat{F}^a\, e^{\frac{i}{\hbar}S_{\rm NCYM}\left[B_\mu+\hbar^{\frac{1}{2}}Q_\mu\right]+iS_{\rm gf}\left[B_\mu,Q_\mu,\hat{F},\hat{\bar{C}},C\right]}, \tag{2.34}$$

where $S_{\rm gf}\big[B_\mu, Q_\mu, \hat{F}, \hat{\bar{C}}, C\big]$ is the gauge-fixing term, which is BRS-exact — thus benefits the BRS quantization method:

$$S_{\rm gf}\big[B_\mu, Q_\mu, \hat{F}, \bar{C}, C\big] = \delta_{\rm BRS}\, X_{\rm gf}\big[B_\mu, Q_\mu, \hat{F}, \hat{\bar{C}}, C\big].$$

Here $\delta_{\rm BRS}$ is the ordinary BRS operator which acts on the fields B_μ, Q_μ as defined in (2.21) and on $\hat{\bar{C}}$ and $\hat{F}$ as defined in (2.32). The $X_{\rm gf}\left[B_\mu, Q_\mu, \hat{F}, \hat{\bar{C}}, C\right]$ is an arbitrary functional — with ghost number -1 — of the fields, which can be expressed as formal series of the fields.

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Taking into account that $\delta^2_{\rm BRS} = 0$, when acting on B_μ, Q_μ, C, $\hat{\bar{C}}$ and $\hat{F}$, respectively, one concludes that $\delta_{\rm BRS} S_{\rm gf} = 0$. Hence, the results presented in [24] also apply here, so that $\Gamma_{\rm DeW}\big[B_\mu\big]$ does not depend on the $X_{\rm gf}\big[B_\mu, Q_\mu, \hat{F}, \bar{C}, C\big]$ that one chooses. For instance, one may choose the standard background field gauge of the ordinary fields, i.e.,

$$X_{gf} = \frac{\hbar^{\frac{1}{2}}}{g^2} \int {\rm tr}\, \bar{C} \left(\alpha \hat{F} + D_\mu \Big[A_\mu [B_\mu + \hbar^{\frac{1}{2}} Q_\mu] \Big] C \right),$$

but this gauge-fixing term will not suit our purpose. We shall choose the following term

$$S_{\rm gf} = \delta_{\rm BRS} \frac{\hbar^{\frac{1}{2}}}{g^2} \int \hat{\bar{C}} \left(\alpha \hat{F} + \hat{D}_\mu \Big[\hat{A}_\mu [B_\mu + \hbar^{\frac{1}{2}} Q_\mu] \Big] \hat{C}[B_\mu, Q_\mu, C] \right), \tag{2.35}$$

instead. Note that $\hat{D}_\mu \left[\hat{A}_\mu \left[B_\mu + \hbar^{\frac{1}{2}} Q_\mu \right] \right]$ is the noncommutative covariant derivative. Now, taking into account (2.21) and (2.32), one concludes that our choice of gauge-fixing term reads

$$S_{\rm gf} = \frac{1}{g^2} \int {\rm tr} \left(\alpha \hat{F}^2 + \hat{F} \hat{D}_\mu [\hat{B}_\mu] \hat{Q}^\mu - \hat{\bar{C}} \hat{D}_\mu [\hat{B}_\mu] \hat{D}^\mu [\hat{B}_\mu + \hbar^{\frac{1}{2}} \hat{Q}_\mu] \hat{C} \right), \tag{2.36}$$

which is the gauge-fixing term corresponding to the noncommutative background field gauge.

3 Establishing quantum equivalence by changing variables in the path integral

Let $\hat{Q}^a_\mu = tr(\hat{Q}_\mu T^a)$ and $\hat{C}^a = tr(\hat{C}\, C^a)$, where $\hat{Q}_\mu$ and $\hat{C}$ are given by the Seiberg-Witten map in (2.17) and (2.18), respectively. Let $J_1[B^a, Q^a]$ and $J_2[B^a, Q^a]$ be the following Jacobian determinants

$$\begin{aligned} J_1[B^a, Q^a] &= \det \frac{\delta \hat{Q}^a_\mu(x)}{\delta Q^b_\nu(y)} = \exp\, {\rm Tr}\, \ln \left(\frac{\delta \hat{Q}^a_\mu(x)}{\delta Q^b_\nu(y)} \right), \\ J_2[B^a, Q^a] &= \det \frac{\delta \hat{C}^a(x)}{\delta C^b(y)} = \exp\, {\rm Tr}\, \ln \left(\frac{\delta \hat{C}^a(x)}{\delta C^b(y)} \right). \end{aligned} \tag{3.1}$$

By changing variables in the path integral in (2.34): $C^a \to \hat{C}^a$ and $Q^a_\mu \to \hat{Q}^a_\mu$, we obtain

$$\begin{aligned} e^{\frac{i}{\hbar}\Gamma_{\rm DeW}[B_\mu]} = \int d\hat{Q}^a_\mu d\hat{C}^a d\hat{\bar{C}}^a d\hat{F}^a \Big\{ & J_1^{-1}[B, Q]\, J_2[B, Q] \\ & \cdot e^{\frac{i}{\hbar} S_{\rm NCYM}\left[\hat{B}_\mu + \hbar^{\frac{1}{2}} \hat{Q}_\mu\right] + i S_{\rm gf}\left[\hat{B}_\mu, \hat{Q}_\mu, \hat{F}, \hat{\bar{C}}, \hat{C}\right]} \Big\}, \end{aligned} \tag{3.2}$$

where $\hat{B}_\mu$ and $\hat{Q}_\mu$ are expressed in terms of B_μ and Q_μ and

$$S_{\rm gf}\big[\hat{B}_\mu, \hat{Q}_\mu, \hat{F}, \hat{\bar{C}}, \hat{C}\big] = S_{\rm BFG}\big[\hat{B}_\mu, \hat{Q}_\mu, \hat{F}, \hat{\bar{C}}, \hat{C}\big].$$

To continue we start with the following *proposition*:

If $J_1[B,Q]=1$ and $J_2[B,Q]=1$, then the right hand side of (3.2) equals the right hand side of (2.1), leading to (1.1), that is: $\Gamma_{\rm DeW}\left[B_\mu\right]=\hat{\Gamma}_{\rm DeW}\left[\hat{B}_\mu[B_\mu]\right]$.

This result is valid on-shell since $\hat{B}_\mu[B_\mu]$ satisfies the noncommutative Yang-Mills equations (2.33), and the reason is the on-shell uniqueness of DeWitt effective action [19, 24].

In summary, if we are able to show that *proposition* holds, that would prove that both theories defined in terms of noncommutative fields and in terms of ordinary fields, through the Seiberg-Witten map, have the same on-shell DeWitt effective action and, therefore, they are dual — i.e., they are different descriptions of the same underlying theory — to each other at the quantum level. We shall show below that, indeed, in dimensional regularization, and in the perturbative regime defined by the coupling constant, our *proposition* holds.

3.1 No one-loop two-point contribution coming from $J_1\left[B,Q\right]$ or $J_2\left[B,Q\right]$

Before we plunge into the general proof that *proposition* holds, we shall show that by employing dimensional regularization the one-loop two-point contribution to

$$\ln\, J_1[B,Q]={\rm Tr}\,\ln\left(\frac{\delta\hat{Q}^a_\mu(x)}{\delta Q^b_\nu(y)}\right), \tag{3.3}$$

vanishes. By working out this simple instance, we shall acquaint ourselves with the techniques that we shall employ in the general case, as well as the type of dimensionally regularized integrals one has to face.

From (2.17) and

$$\frac{\delta Q^a_\mu(p)}{\delta Q^b_\nu(y)}=e^{-ipy}\delta^a_b\delta^\nu_\mu, \tag{3.4}$$

one obtains

$$\begin{aligned}\frac{\delta\hat{Q}^a_\mu(x)}{\delta Q^b_\nu(y)}&=\delta^a_b\delta^\nu_\mu\delta(x-y)+\frac{\delta}{\delta Q^b_\nu(y)}\left\{\sum_{n=2}^{\infty}\int\prod_{i=1}^{n}\frac{d^4p_i}{(2\pi)^4}e^{i\left(\sum\limits_{i=1}^{n}p_i\right)x}\right.\\&\qquad\cdot n\,{\rm tr}\left(T^a\mathfrak{A}^{(n)}_\mu\left[(a_1,\mu_1,p_1),\dots,(a_n,\mu_n,p_n);\theta\right]\right)\\&\qquad\left.\cdot\,\tilde{B}^{a_1}_{\mu_1}(p_1)\dots\tilde{B}^{a_{n-1}}_{\mu_{n-1}}(p_{n-1})\tilde{Q}^{a_n}_{\mu_n}(p_n)\right\}+O(\hbar^{\frac{1}{2}})\\&=\delta^a_b\delta^\mu_\nu\delta(x-y)+\mathcal{M}_1\left[B\right]^{a\nu}_{b\mu}(x,y)+\mathcal{M}_2\left[B\right]^{a\nu}_{b\mu}(x,y),+O(B^3)+O(\hbar^{\frac{1}{2}}),\end{aligned} \tag{3.5}$$

with

$$\begin{aligned}\mathcal{M}_1\left[B\right]^{a\nu}_{b\mu}(x,y)&=2\int\frac{d^4p_1}{(2\pi)^4}e^{ip_1x}e^{-ip_2(x-y)}\,{\rm tr}\left(T^a\mathfrak{A}^{(2)}_\mu\left[(a_1,\mu_1,p_1);(b,\nu,p_2);\theta\right]\right)\tilde{B}^{a_1}_{\mu_1}(p_1)\,,\\\mathcal{M}_2\left[B\right]^{a\nu}_{b\mu}(x,y)&=3\int\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}e^{i(p_1+p_2)x}e^{ip_3(x-y)}\\&\qquad\cdot{\rm tr}\left(T^a\mathfrak{A}^{(3)}_\mu\left[(a_1,\mu_1,p_1),(a_2,\mu_2,p_2);(b,\nu,p_3);\theta\right]\right)\tilde{B}^{a_1}_{\mu_1}(p_1)\tilde{B}^{a_2}_{\mu_2}(p_2).\end{aligned} \tag{3.6}$$

Now, substituting the previous results in (3.3), one gets

$$
\begin{aligned}
&\operatorname{Tr}\ln\left(\frac{\delta\hat{Q}^a_\mu(x)}{\delta Q^b_\nu(y)}\right)=\operatorname{Tr}\ln\left(\delta^a_b\delta^\mu_\nu\delta(x-y)+\mathcal{M}_1\,[B]^{a\nu}_{b\mu}\,(x,y)+\mathcal{M}_2\,[B]^{a\nu}_{b\mu}\,(x,y)\right)+O(B^3)+O(\hbar^{\frac{1}{2}})\\
&=\operatorname{Tr}\ln\left(\mathbb{1}+\mathcal{M}_1+\mathcal{M}_2\right)+O(B^3)+O(\hbar^{\frac{1}{2}})=\operatorname{Tr}\sum_{k=1}^{\infty}\frac{(-)^{k+1}}{k}(\mathcal{M}_1+\mathcal{M}_2)^k+O(B^3)+O(\hbar^{\frac{1}{2}})\\
&=\operatorname{Tr}\mathcal{M}_1+\operatorname{Tr}\mathcal{M}_2+\operatorname{Tr}\mathcal{M}_1\,\mathcal{M}_1+O(B^3)+O(\hbar^{\frac{1}{2}}),
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
\operatorname{Tr}\mathcal{M}_1&=\int d^4x\,\mathcal{M}_1\,[B]^{a\mu}_{a\mu}\,(x,x)\\
&=\int\frac{d^{2\omega}p_1}{(2\pi)^{2\omega}}(2\pi)^{2\omega}\delta(p_1)B^{a_1}_{\mu_1}(p_1)\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\operatorname{tr}\left(T^a\mathfrak{A}^{(2)}_\mu\,[(a_1,\mu_1,p_1),(a,\mu,q);\theta]\right),
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
\operatorname{Tr}\mathcal{M}_2=\int d^4x\,\mathcal{M}_2\,[B]^{a\mu}_{a\mu}\,(x,x)=&\int\frac{d^{2\omega}p_1}{(2\pi)^{2\omega}}\int\frac{d^{2\omega}p_2}{(2\pi)^{2\omega}}(2\pi)^{2\omega}\Big\{\delta(p_1+p_2)B^{a_1}_{\mu_1}(p_1)B^{a_2}_{\mu_2}(p_2)\\
&\cdot 3\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\operatorname{tr}\left(T^a\mathfrak{A}^{(3)}_\mu\,[(a_1,\mu_1,p_1),(a_2,\mu_2,p_2),(a,\mu,q);\theta]\right)\Big\},
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
\operatorname{Tr}\mathcal{M}_1\mathcal{M}_1=&\int d^{2\omega}x\int d^4y\,\mathcal{M}_1\,(B)^{a\mu}_{a'\mu'}\,(x,y)\mathcal{M}_1\,(B)^{a'\mu'}_{a\mu}\,(y,x)\\
=&\int\frac{d^{2\omega}p_1}{(2\pi)^{2\omega}}\int\frac{d^{2\omega}q_1}{(2\pi)^{2\omega}}(2\pi)^{2\omega}\Big\{\delta(p_1+q_1)\tilde{B}^{a_1}_{\mu_1}(p_1)\tilde{B}^{b_1}_{\nu_1}(q_1)\\
&\cdot 4\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\operatorname{tr}\left(T^a\mathfrak{A}^{(2)}_{\mu_2}\,[(a_1,\mu_1,p_1),(a_2,\mu,q);\theta]\right)\\
&\cdot\operatorname{tr}\left(T^{a_2}\mathfrak{A}^{(2)}_\mu\,[(b_1,\nu_1,p_1),(a,\mu_2,p_1+q);\theta]\right)\Big\}.
\end{aligned}
\tag{3.10}
$$

Let us show now that in dimensional regularization $\operatorname{Tr}\mathcal{M}_1=0$. The term $\mathfrak{A}^{(2)}_\mu\,[(a_1,\mu_1,p_1),(a,\mu,q);\theta]$ can be obtained from $\mathbb{A}^{(2)}_\mu$ in section III of ref. [25]:

$$
\begin{aligned}
\mathfrak{A}^{(2)}_\mu\,[(a_1,\mu_1,p_1),(a_2,\mu_2,p_2);\theta]=\frac{1}{2}\Big(&\mathbb{A}^{(2)}_\mu\,[(a_1,\mu_1,-p_1),(a_2,\mu_2,-p_2);\theta]\\
&+\mathbb{A}^{(2)}_\mu\,[(a_2,\mu_2,-p_2),(a_1,\mu_1,-p_1);\theta]\Big).
\end{aligned}
\tag{3.11}
$$

Hence, in dimensional degularization the loop integral — the integral over q — (in $\operatorname{Tr}\mathcal{M}_1$ — see (3.8) —) reads

$$
\begin{aligned}
&\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\operatorname{tr}\left(T^a\mathfrak{A}^{(2)}_\mu\,[(a_1,\mu_1,p_1),(a,\mu,q));\theta]\right)\\
&=\frac{1}{2}\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\operatorname{tr}\left(\mathbb{A}^{(2)}_\mu\,[(a_1,\mu_1,-p_1),(a,\mu,q);\theta]+\mathbb{A}^{(2)}_\mu\,[(a,\mu,q),(a_1,\mu_1,-p_1);\theta]\right)\\
&=-\frac{1}{4}\operatorname{tr}\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\theta^{ij}(2q_j\delta^{\mu_1}_i\delta^\mu_\mu-q_\mu\delta^{\mu_1}_i\delta^\mu_j)\left(T^aT^{a_1}T^a\frac{e^{-\frac{i}{2}q\theta p}-1}{q\theta p}-T^aT^aT^{a_1}\frac{e^{\frac{i}{2}q\theta p}-1}{q\theta p}\right)\\
&=0,
\end{aligned}
\tag{3.12}
$$

since

$$\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \frac{q_{\mu_1}\dots q_{\mu_r}}{q\theta p} = 0, \quad \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} q_{\mu_1}\dots q_{\mu_r} \frac{e^{i\xi q\theta p}}{q\theta p} = 0, \forall \xi. \tag{3.13}$$

One may actually use $\delta(p_1)$ to further simplify the argument, as only the second vanishing identity above would be needed. We have included the discussion of the vanishing of the previous type of integrals due to the employment of dimensional regularization in the appendix B.

Next, by integrating out the Dirac delta function, $\delta(p_1+p_2)$ in (3.9), one comes to the conclusion that to work out $\mathrm{Tr}\mathcal{M}_2$, one has to compute the following dimensionally regularized integral

$$\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \mathrm{tr}\left(T^a \mathfrak{A}^{(3)}_\mu \left[(a_1,\mu_1,p_1),(a_2,\mu_2,-p_1),(a,\mu,q);\theta\right]\right), \tag{3.14}$$

where $\mathfrak{A}^{(3)}_\mu \left[(a_1,\mu_1,p_1),(a_2,\mu_2,-p_1),(a,\mu,q);\theta\right]$ is obtained from $\mathbb{A}^{(3)}_\mu$ in equation (3.1) of ref. [25] by appropriate symmetrization. By expressing $\mathfrak{A}^{(3)}_\mu[(a_1,\mu_1,p_1),(a_2,\mu_2,-p_1),(a,\mu,q);\theta]$ in terms of $\mathbb{A}^{(3)}_\mu$, one concludes that the integral in (3.14) is a linear combination of the following types of dimensionally regularized integrals:

$$\begin{aligned}
&\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\mathbb{Q}(q)\,\mathbb{I}(-p_1,p_1,q,\theta), && \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\mathbb{Q}(q)\,\mathbb{I}(-p_1,-q,p_1,\theta),\\
&\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\mathbb{Q}(q)\,\mathbb{I}(-q,-p_1,p_1,\theta), && \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\mathbb{Q}(q)\,\mathbb{I}(p_1,p_1,-q,\theta),\\
&\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\mathbb{Q}(q)\,\mathbb{I}(p_1,-q,p_1,\theta), && \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\mathbb{Q}(q)\,\mathbb{I}(-q,p_1,p_1,\theta)\\
&\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\mathbb{Q}(q)\,\mathbb{F}(-p_1,p_1,q,\theta), && \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\mathbb{Q}(q)\,\mathbb{F}(-p_1,-q,p_1,\theta),\\
&\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\mathbb{Q}(q)\,\mathbb{F}(-q,-p_1,p_1,\theta), && \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\mathbb{Q}(q)\,\mathbb{F}(p_1,p_1,-q,\theta),\\
&\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\mathbb{Q}(q)\,\mathbb{F}(p_1,-q,p_1,\theta), && \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\mathbb{Q}(q)\,\mathbb{F}(-q,p_1,p_1,\theta);
\end{aligned}$$

where $\mathbb{Q}(q)$ denotes symbolically a monomial in q (i.e. $\mathbb{Q}\equiv q_{\mu_1}\dots q_{\mu_r}$), and

$$\mathbb{I}(p_1,p_2,p_3,\theta) = (p_2\theta p_3)^{-1}\left[\frac{e^{-\frac{i}{2}(p_1\theta p_2+p_1\theta p_3+p_2\theta p_3)}-1}{p_1\theta p_2+p_1\theta p_3+p_2\theta p_3} - \frac{e^{-\frac{i}{2}p_1\theta(p_2+p_3)}-1}{p_1\theta(p_2+p_3)}\right], \tag{3.15}$$

$$\mathbb{F}(p_1,p_2,p_3,\theta) = \frac{e^{-\frac{i}{2}(p_1\theta p_2+p_1\theta p_3+p_2\theta p_3)}-1}{p_1\theta p_2+p_1\theta p_3+p_2\theta p_3}. \tag{3.16}$$

Hence

$$\mathbb{I}(-p_1,p_1,q,\theta)=\frac{e^{\frac{i}{2}q\theta p_1}-1}{(q\theta p_1)^2}-\frac{i}{2q\theta p_1},\qquad \mathbb{I}(-p_1,-q,p_1,\theta)=\frac{(e^{\frac{i}{2}q\theta p_1}-1)^2}{2(q\theta p_1)^2},$$
$$\mathbb{I}(-q,-p_1,p_1,\theta)=-\frac{1}{8},\qquad \mathbb{I}(p_1,p_1,-q,\theta)=\frac{(e^{-\frac{i}{2}q\theta p_1}-1)^2}{2(q\theta p_1)^2},$$
$$\mathbb{I}(p_1,-q,p_1,\theta)=\frac{e^{-\frac{i}{2}q\theta p_1}-1}{(q\theta p_1)^2}+\frac{i}{2q\theta p_1},\qquad \mathbb{I}(q,-p_1,p_1,\theta)=-\frac{1}{8},$$
$$\mathbb{F}(-p_1,p_1,q,\theta)=-\frac{i}{2},\qquad \mathbb{F}(-p_1,-q,p_1,\theta)=\frac{1-e^{iq\theta p_1}}{2q\theta p_1},$$
$$\mathbb{F}(-q,-p_1,p_1,\theta)=-\frac{i}{2},\qquad \mathbb{F}(p_1,p_1,-q,\theta)=\frac{e^{-iq\theta p_1}-1}{2q\theta p_1},$$
$$\mathbb{F}(p_1,-q,p_1,\theta)=-\frac{i}{2},\qquad \mathbb{F}(q,-p_1,p_1,\theta)=-\frac{i}{2}.$$

Putting it all together one reaches the conclusion that all the integrals listed above are of the type

$$\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\frac{e^{iq\theta p}}{(q\theta p_1)^{n_1}\dots(q\theta p_r)^{n_r}},\tag{3.17}$$

which vanish — see appendix B for details — in dimensional regularization. We have thus shown that $\mathrm{Tr}\,\mathcal{M}_2=0$, in dimensional regularization.

Let us finally show that $\mathrm{Tr}\,\mathcal{M}_1\mathcal{M}_1=0$ in dimensional regularization. The loop integral over q, contributing to $\mathrm{Tr}\,\mathcal{M}_1\mathcal{M}_1$ — as seen from (3.10) — is:

$$\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\mathrm{tr}\left(T^a\mathfrak{A}^{(2)}_{\mu_2}\left[(a_1,\mu_1,p_1),(a_2,\mu,q);\theta\right]\right)\mathrm{tr}\left(T^{a_2}\mathfrak{A}^{(2)}_{\mu}\left[(b_1,\nu_1,p_1),(a,\mu_2,p_1+q);\theta\right]\right);\tag{3.18}$$

but this integral vanishes since it is, again, a linear combination of integrals of the type

$$\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\mathbb{Q}\left(\frac{e^{\frac{i}{2}q\theta p_1}-1}{q\theta p_1}\right)\left(\frac{e^{\pm\frac{i}{2}q\theta p_1}-1}{q\theta p_1}\right).\tag{3.19}$$

However, these integrals — appendix B — are equal to zero in dimensional regularization.

In summary, we have just shown that in dimensional regularization $\mathrm{Tr}\,\mathcal{M}_1=0$, $\mathrm{Tr}\,\mathcal{M}_2=0$ and $\mathrm{Tr}\,\mathcal{M}_1\mathcal{M}_1=0$, and, hence — see (3.7) and (3.3) — one obtains

$$\ln J_1[B,Q]=0+O(B^3)+O(\hbar^{\frac{1}{2}}).$$

Finally, since the same types of integral contribute to $J_2[B,Q]$ it is plain that

$$\ln J_2[B,Q]=0+O(B^3)+O(\hbar^{\frac{1}{2}}),$$

also holds, and therefore, the one-loop two-point contribution to $\Gamma_{\mathrm{DeW}}[B_\mu]$ does not receive contributions neither from $J_1[B,Q]$ nor from $J_2[B,Q]$.

Later in this paper a head-on — i.e., by using the Feynman rules for the ordinary fields and not changing variables in the path integral — computation of the same two-point function will be performed. We are now ready to show that there are no nontrivial contribution either to $J_1[B,Q]$ or to $J_2[B,Q]$.

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3.2 Triviality of the full Jacobian determinants

It is shown in the appendix B that

$$\frac{\delta\hat{Q}^a_\mu(x)}{\delta Q^b_\nu(y)} = \frac{1}{\hbar^{\frac{1}{2}}}\frac{\delta\hat{A}^a_\mu(x)}{\delta Q^b_\nu(y)} \tag{3.20}$$
$$= \delta^a_b\delta^\nu_\mu\delta(x-y)+\sum_{n=2}^{\infty}\int\prod_{i=1}^{n}\frac{d^4p_i}{(2\pi)^4}e^{i\left(\sum_{i=1}^{n-1}p_i\right)x}\,e^{ip_n(x-y)}\mathcal{M}^{(n)\,a\,\nu}_{\,b\,\mu}(p_1,p_2,\dots p_{n-1};p_n;\theta),$$

where

$$\mathcal{M}^{(n)\,a\,\nu}_{\,b\,\mu}(p_1,p_2,\dots p_{n-1};p_n;\theta) \tag{3.21}$$
$$= n\,\mathrm{tr}\Big(T^a\mathfrak{A}^{(n)}_\mu\left[(a_1,\mu_1,p_1),\dots,(a_{n-1},\mu_{n-1},p_{n-1}),(b,\nu,p_n);\theta\right]\Big)\tilde{A}^{a_1}_{\mu_1}(p_1)\dots\tilde{A}^{a_{n-1}}_{\mu_{n-1}}(p_{n-1}).$$

Note that the definition of splitting in the momentum space reads $\tilde{A}^{a_i}_{\mu_i}(p_i) = \tilde{B}^{a_i}_{\mu_i}(p_i) + \hbar^{\frac{1}{2}}\tilde{Q}^{a_i}_{\mu_i}(p_i)$ for all i. Now let's first define total momenta l_i, $i = 1,\dots,m+1$, as the following sums

$$l_1 = \sum_{i_1=1}^{n_1-1}p_{1,i_1},\;\; l_2 = \sum_{i_2=1}^{n_2-1}p_{2,i_2},\dots,\;\; l_m = \sum_{i_m=1}^{n_m}p_{m,i_m},\;\; l_{m+1} = \sum_{i_{m+1}=1}^{n_{m+1}}p_{m+1,i_{m+1}},$$

then, by taking into account (3.20) and carrying out a lengthy straightforward computation — see appendix B for details — one gets

$$\ln\, J_1[B,Q] = \mathrm{Tr}\ln\left(\frac{\delta\hat{Q}^a_\mu(x)}{\delta Q^b_\nu(y)}\right)$$
$$= \sum_{n=2}^{\infty}\int\prod_{i=1}^{n-1}\frac{d^4p_i}{(2\pi)^4}\,\delta\left(\sum_{i=1}^{n-1}p_i\right)\int\frac{d^4q}{(2\pi)^4}\,\mathcal{M}^{(n)\,a\,\mu}_{\,a\,\mu}(p_1,p_2,\dots,p_{n-1};q;\theta)$$
$$+\sum_{m=1}^{\infty}\frac{(-1)^m}{m+1}\sum_{n_1=2}^{\infty}\cdots\sum_{n_{m+1}=2}^{\infty}\int\prod_{i_1=1}^{n_1-1}\frac{d^4p_{1,i_1}}{(2\pi)^4}\cdots\int\prod_{i_{m+1}=1}^{n_{m+1}-1}\frac{d^4p_{m+1,i_{m+1}}}{(2\pi)^4}\,\delta\left(\sum_{i=1}^{m+1}l_i\right)$$
$$\int\frac{d^4q}{(2\pi)^4}\Bigg[\mathcal{M}^{(n_1)\,a\,\mu_1}_{\,a_1\,\mu}(p_{1,1},p_{1,2},\dots,p_{1,n_1-1};q;\theta)\cdot\mathcal{M}^{(n_2)\,a_1\,\mu_2}_{\,a_2\,\mu_1}(p_{2,1},p_{2,2},\dots,p_{2,n_2-1};q-l_2;\theta)$$
$$\cdot\mathcal{M}^{(n_3)\,a_2\,\mu_3}_{\,a_3\,\mu_2}(p_{3,1},p_{3,2},\dots,p_{3,n_3-1};q-l_2-l_3;\theta)$$
$$\dots$$
$$\cdot\mathcal{M}^{(n_m)\,a_{m-1}\,\mu_m}_{\,a_m\quad\mu_{m-1}}\left(p_{m,1},p_{m,2},\dots,p_{m,n_m-1};q-\sum_{i=2}^{m}l_i;\theta\right)$$
$$\cdot\mathcal{M}^{(n_{m+1})\,a_m\,\mu}_{\phantom{(n_{m+1})}\,a\;\;\mu_m}\left(p_{m+1,1},p_{m+1,2},\dots,p_{m+1,n_{m+1}-1};q-\sum_{i=2}^{m+1}l_i;\theta\right)\Bigg]. \tag{3.22}$$

The general structure of the master integral (3.22) above can be visualized by a 1-loop diagram, as given in figure 1.

Now, in a view of previous equations (3.21) and (3.22), to complete computation of $\ln J_1[B,Q]$, one has to work out the following dimensionally regularized type of integrals over the internal momenta q^μ:

$$\begin{aligned}\mathfrak{V} = \int \frac{d^Dq}{(2\pi)^D}\Bigg\{&\mathrm{tr}\Big(T^a\mathfrak{A}^{(n_1)}_\mu\left[(b_{1,1},\nu_{1,1},p_{1,1}),\ldots,(b_{1,n_1-1},\nu_{1,n_1-1},p_{1,n_1-1}),(a_1,\mu_1,q);\theta\right]\Big)\\
&\cdot\mathrm{tr}\Big(T^{a_1}\mathfrak{A}^{(n_2)}_{\mu_1}\left[(b_{2,1},\nu_{2,1},p_{2,1}),\ldots,(b_{2,n_2-1},\nu_{2,n_2-1},p_{2,n_2-1}),(a_2,\mu_2,q-l_2);\theta\right]\Big)\\
&\cdot\mathrm{tr}\Big(T^{a_2}\mathfrak{A}^{(n_3)}{}_{\mu_2}\left[(b_{3,1},\nu_{3,1},p_{3,1}),\ldots,(b_{3,n_3-1},\nu_{3,n_3-1},p_{3,n_3-1}),(a_3,\mu_3,q-l_2-l_3);\theta\right]\Big)\\
&\cdots\\
&\cdot\mathrm{tr}\Bigg(T^{a_{m-1}}\mathfrak{A}^{(n_m)}{}_{\mu_{m-1}}\Bigg[(b_{m,1},\nu_{m,1},p_{m,1}),\ldots,\\
&\qquad b_{m,n_m-1},\nu_{m,n_m-1},p_{m,n_m-1}),\left(a_m,\mu_m,q-\sum_{i=2}^m l_i\right);\theta\Bigg]\Bigg)\\
&\cdot\mathrm{tr}\Bigg(T^{a_m}\mathfrak{A}^{(n_{m+1})}{}_{\mu_m}\Bigg[(b_{m+1,1},\nu_{m+1,1},p_{m+1,1}),\ldots,\\
&\qquad (b_{m+1,n_{m+1}-1},\nu_{m+1,n_{m+1}-1},p_{m+1,n_{m+1}-1}),\left(a,\mu,q-\sum_{i=2}^{m+1} l_i\right);\theta\Bigg]\Bigg)\Bigg\}.\end{aligned} \tag{3.23}$$

However, according to the discussion at the end of subsection 2.2.1, the previous integral is a linear combination of integrals of the type

$$\mathfrak{I} = \int \frac{d^Dq}{(2\pi)^D}\,\mathbb{Q}(q)\,\mathbb{I}(q\theta k_i, k_i\theta k_j), \tag{3.24}$$

where $\mathbb{Q}(q) = q^{\rho_1}q^{\rho_2}\cdots q^{\rho_n}$, $q\theta k_i = q_\mu\theta^{\mu\nu}k_{i\nu}$, $i = 1,\ldots,s$ and $k_i\theta k_j = k_{i\mu}\theta^{\mu\nu}k_{j\nu}$, $i,j = 1,\ldots,s$. Here n and s run over all relevant momenta other than q, in general. It is important to stress that $\mathbb{Q}(q)$ is a monomial on q^ρ and that the function $\mathbb{I}$ in the integrand of the previous integral is a function of the variables $q\theta k_i$ and $k_i\theta k_j$ only, and, hence, as shown in the appendix C, one concludes that

$$\mathfrak{I} = 0 \quad \text{and} \quad \mathfrak{V} = 0. \tag{3.25}$$

By substituting $\mathfrak{V} = 0$ in (3.22), obtains that in dimensional regularization the following logarithm vanishes: $\ln J_1[B,Q] = 0$, and

$$J_1[B,Q] = 1. \tag{3.26}$$

It is plain that identical lines of arguments apply to $J_2[B,Q]$ as well. Thus as expected, our *proposition* has been proven.

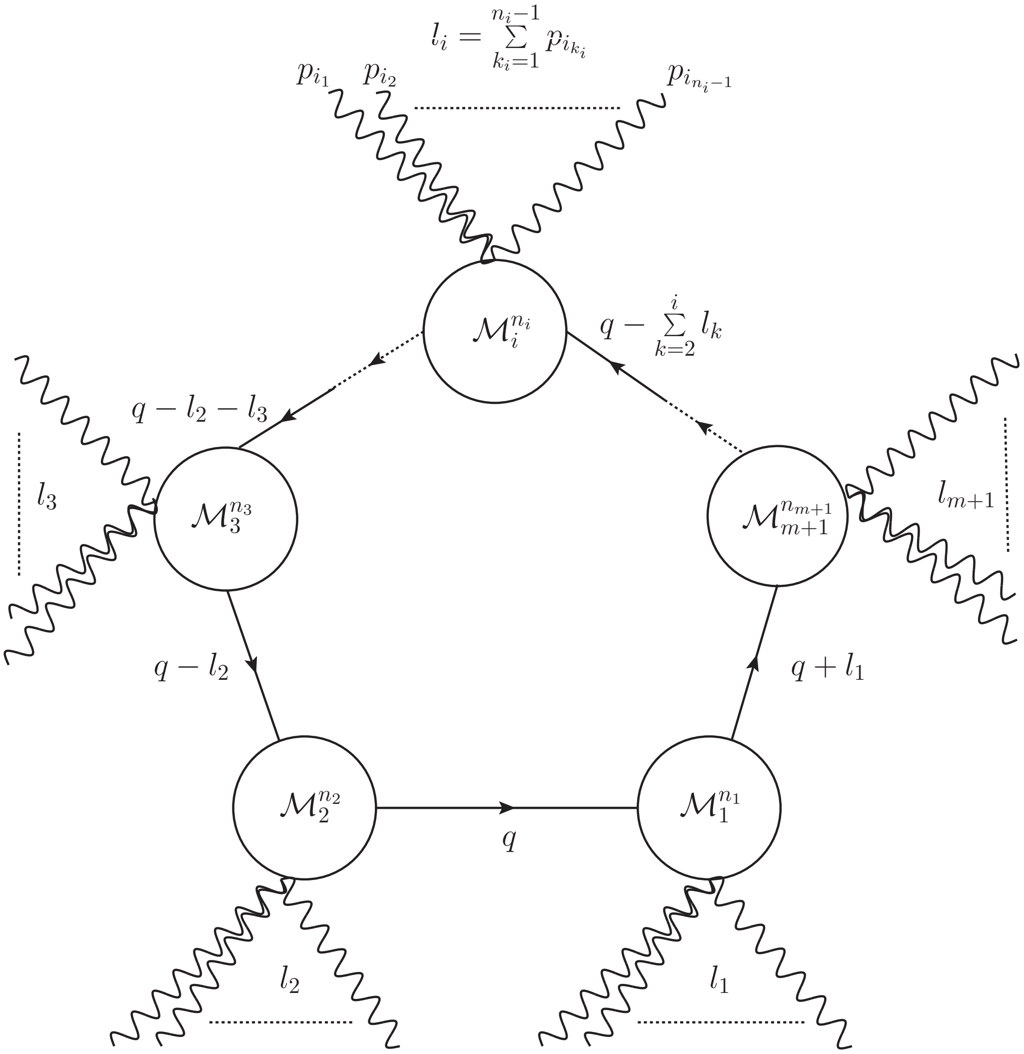

Figure 1. The one-loop diagram interpretation/ilustration of (3.22): each circle corresponds one $\mathcal{M}^{(n_i)}$, wavy lines denote the gauge field operators, either background or quantum, within the $\mathcal{M}^{(n_i)}$. The l_i's are then just the total momentum brought in by these field operators. The solid line flows in each circle gives the assignment of $q - \sum\limits_k l_k$ into the corresponding $\mathcal{M}^{(n_i)}$ in (3.22).

3.3 Incorporating adjoint matter

The θ-exact Seiberg-Witten map for scalar or fermion fields in the adjoint reads — see [26]:

$$\begin{aligned}
\hat{\Phi}\left[A_\mu, \Phi, \theta\right](x) &= \Phi(x) + \sum_{n=1}^{\infty} \mathcal{F}^{(n)}(x), \\
\mathcal{F}^{(n)}(x) &= \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} e^{i\left(p+\sum\limits_{i=1}^{n} p_i\right)x} \mathfrak{F}^{(n)}\left[(a_1, \mu_1, p_1), \ldots, (a_n, \mu_n, p_n); (a, p); \theta\right] \\
&\quad \cdot \tilde{A}_{\mu_1}^{a_1}(p_1) \ldots \tilde{A}_{\mu_n}^{a_n}(p_n) \Phi^a(p),
\end{aligned} \tag{3.27}$$

where $\Phi = \Phi^a T^a$ denotes an ordinary scalar or fermion field transforming under the adjoint of U(N). Now, taking into account the recursive equations — see section III of ref. [26] — which yield $\hat{\Phi}\left[A_\mu, \Phi, \theta\right](x)$ have similar θ and momentum structure to the one for $\hat{A}_\mu\left[A_\mu, \theta\right]$, it is easy to see that $\mathfrak{F}^{(n)}\left[(a_1, \mu_1, p_1), \ldots, (a_n, \mu_n, p_n); (a, p); \theta\right]$ is also a linear combination of functions of the type in (2.24). Hence, one also concludes that the Jacobian determinant for the change of variables $\Phi^a \to \hat{\Phi}^a(x) = \operatorname{tr}(\hat{\Phi}(x) T^a) = \operatorname{tr}(T^a \hat{\Phi}\left[A_\mu, \Phi, \theta\right](x))$ in the path integral over the fields Φ^a is one, i.e.:

$$\det \frac{\delta \hat{\Phi}^a(x)}{\delta \Phi^a(y)} = 1. \tag{3.28}$$

Hence the inclusion of matter fields in the adjoint — and for that matter any type of matter fields — does not change the conclusion that we have reached above for gauge fields, i.e., that the θ-exact Seiberg-Witten map associates every quantum field theory, with gauge group U(N) and formulated in terms of noncommutative fields, to an ordinary gauge theory, with gauge group U(N), which is dual to the former at the quantum level, because, indeed, they have the same on-shell DeWitt effective action.

Note that the massless tadpole integrals also vanish in the dimensional-reduction scheme, which preserves supersymmetry manifestly in the one loop. Therefore our conclusion here should be also valid for the noncommutative supersymmetric Yang-Mills theories (NCSYMs).

4 Testing the quantum equivalence by direct computation: the one-loop two-point function

We choose to test the formal equivalence established in the last section by computing explicitly one-loop quantum correction to the quadratic part of the effective action in the noncommutative U(1) gauge theory prior to and after the Seiberg-Witten map. At this specific order, the general equivalence relation (1.1) reduces to a much simpler relation

$$\begin{aligned}\int \frac{d^D p}{(2\pi)^D} \tilde{B}_\mu(-p) \Gamma^{\mu\nu}(p) \tilde{B}_\nu(p) &= \int \frac{d^D p}{(2\pi)^D} \tilde{\hat{B}}_\mu[\tilde{B}_\mu(-p)] \hat{\Gamma}^{\mu\nu}(p) \tilde{\hat{B}}_\nu[\tilde{B}_\mu(p)] \\ &= \int \frac{d^D p}{(2\pi)^D} \tilde{B}_\mu(-p) \hat{\Gamma}^{\mu\nu}(p) \tilde{B}_\nu(p)\end{aligned} \tag{4.1}$$

when $\tilde{B}_\nu(p)$ is placed on-shell, because only the zeroth order of the SW map counts here.

We start by reviewing the standard procedure for computng the DeWitt effective action of U(1) gauge theory perturbatively in the background field formalism/method (BFM) [19, 20], which evaluates all 1-PI diagrams with all background field external legs and all integrand fields $(\hat{Q}_\mu, \hat{\bar{C}}, \hat{C}, \hat{F})$ internal line using the following action $\hat{S}_{\text{loop}}$

$$\hat{S}_{\text{loop}} = S_{\text{gf}} + S_{\text{NCYM}}\left[\hat{B}_\mu + \hat{Q}_\mu\right] - S_{\text{NCYM}}\left[\hat{B}_\mu\right] - \int \frac{\delta S_{\text{NCYM}}\left[\hat{B}_\mu\right]}{\delta \hat{B}_\mu} \hat{Q}_\mu. \tag{4.2}$$

Once the SW map is employed, one may choose to map the action above, making it

$$\begin{aligned}S_{\text{loop}} &= S_{\text{gf}}\big[B_\mu, Q_\mu, \hat{\bar{C}}, C, \hat{F}\big] + S_{\text{NCYM}}\big[\hat{B}_\mu[B_\mu] + \hat{Q}_\mu[Q_\mu, B_\mu]\big] \\ &\quad - S_{\text{NCYM}}\big[\hat{B}_\mu[B_\mu]\big] - \int \frac{\delta S_{\text{NCYM}}\big[\hat{B}_\mu[B_\mu]\big]}{\delta \hat{B}_\mu}[B_\mu, Q_\mu]\hat{Q}_\mu[B_\mu, Q_\mu];\end{aligned} \tag{4.3}$$

or to map the classical gauge-fixed action then subtract the equations of motion with respect to the commutative/ordinary fields, i.e.

$$\begin{aligned}S'_{\text{loop}} &= S_{\text{gf}}[B_\mu, Q_\mu, \hat{\bar{C}}, C, \hat{F}] \\ &\quad + S_{\text{NCYM}}\big[\hat{B}_\mu[B_\mu] + \hat{Q}_\mu[Q_\mu, B_\mu]\big] - S_{\text{NCYM}}\big[\hat{B}_\mu[B_\mu]\big] - \int \frac{\delta S_{\text{NCYM}}\big[\hat{B}_\mu[B_\mu]\big]}{\delta B_\mu} Q_\mu;\end{aligned} \tag{4.4}$$

These two actions are equivalent on-shell as long as the Seiberg-Witten map is invertible, as proven in the appendix A, yet they are but not identical to each other because of the additional field redefinition factor. We choose to proceed with S_{loop} in the computations presented below. As we will see soon, this choice leads to result directly identical to the computation using noncommutative fields, i.e.[2]

$$\hat{\Gamma}^{\mu\nu}(p) = \Gamma^{\mu\nu}(p). \tag{4.5}$$

We are going to use the extended version of dimensional regularization scheme as in [17], which we know to be compatible with the prescriptions used in subsection 3.1. To simplify the computation we choose $\alpha = 1$ and have the auxiliary field F integrated out.

4.1 Model definition

As our first test we choose S_{gf} to be the background field gauge with respect to the noncommutative fields

$$S_{\text{gf}} = S_{\text{BFG}} = \frac{1}{g^2}\int \operatorname{tr} \hat{\delta}_{\text{BRS}}\, \hat{\bar{C}} \left(\alpha \hat{F} + \hat{D}_\mu\big[\hat{B}_\mu\big]\hat{Q}^\mu\right). \tag{4.6}$$

The U(1) theory version of (4.3) then reads

$$\begin{aligned}S_{\text{U(1)}_{\text{loop}}} &= -\frac{1}{4g^2}\int \left(\hat{D}_\mu\big[\hat{B}_\mu\big]\hat{Q}_\nu - \hat{D}_\nu\big[\hat{B}_\mu\big]\hat{Q}_\mu\right)^2 - \frac{i}{2g^2}\int \hat{F}^{\mu\nu}\big[\hat{B}\big]\left[\hat{Q}_\mu \overset{\star}{,} \hat{Q}_\nu\right] \\ &\quad - \frac{i}{2g^2}\int \left(\hat{D}_\mu\big[\hat{B}_\mu\big]\hat{Q}_\nu - \hat{D}_\nu\big[\hat{B}_\mu\big]\hat{Q}_\mu\right)\left[\hat{Q}_\mu \overset{\star}{,} \hat{Q}_\nu\right] + \frac{1}{4g^2}\int \left(\left[\hat{Q}_\mu \overset{\star}{,} \hat{Q}_\nu\right]\right)^2 \\ &\quad - \frac{1}{g^2}\int \left(\frac{1}{2}\left(\hat{D}_\mu\big[\hat{B}_\mu\big]\hat{Q}^\mu\right)^2 + \bar{C}\hat{D}_\mu\big[\hat{B}_\mu\big]\hat{D}^\mu\big[\hat{B}\big]\hat{C}\right).\end{aligned} \tag{4.7}$$

To perform the one-loop computation we must expand this action up to the $BBQQ$ order, which is worked out in details in the appendix D. In the end we get[3]

$$\begin{aligned}S^{(1)}_{\text{U(1)}} &= -\frac{1}{4}\int (\partial_\mu Q_\nu - \partial_\nu Q_\mu)^2 - \frac{1}{2}(\partial_\mu Q^\mu)^2 - \bar{C}\Box C \\ &\quad + S_{BQQ} + S_{BBQQ} + S_{Bc\bar{c}} + S_{BBc\bar{c}} + \mathcal{O}(BBB),\end{aligned} \tag{4.8}$$

[2]Our prior computation in [17] would actually correspond to the same evaluation but with S'_{loop}, which is, because of the proof in the appendix A, equivalent to the results here on-shell.

[3]We assume $g = \hbar = 1$ from now on for simplicity, actually coupling is g for BQQ and g^2 for $BBQQ$. As a convention interactions with subindex 2 are derived from S_{gf}, while those with subindex 1 are from the rest of S_{loop}. We assume $\hat{\bar{C}} \equiv \bar{C}$ from now on, too.

where

$$
\begin{aligned}
S_{BQQ} &= S_{BQQ_1} + S_{BQQ_2}, \\
S_{BQQ_1} &= -\frac{1}{2}\int iB_{\mu\nu}\,[Q^\mu \stackrel{\star}{,} Q^\nu] \\
&\quad + Q_{\mu\nu}\theta^{ij}\left(B_{i\mu} \star_2 Q_{j\nu} + Q_{i\mu} \star_2 B_{j\nu} - B_i \star_2 \partial_j Q_{\mu\nu} - Q_i \star_2 \partial_j B_{\mu\nu}\right), \qquad (4.9)\\
S_{BQQ_2} &= -\int (\partial_\mu Q^\mu)(\partial^\nu \hat{\hat{Q}}^{(1)}_\nu) + i(\partial_\mu Q^\mu)\,[B_\mu \stackrel{\star}{,} Q^\mu]\,, \qquad (4.10)\\
S_{BBQQ} &= S_{BBQQ_1} + S_{BBQQ_2} \qquad (4.11)\\
S_{BBQQ_1} &= -\frac{1}{4}\int \Big(\theta^{ij}\left(B_{i\mu} \star_2 Q_{j\nu} + Q_{i\mu} \star_2 B_{j\nu} - B_i \star_2 \partial_j Q_{\mu\nu} - Q_i \star_2 \partial_j B_{\mu\nu}\right)\Big)^2 \\
&\quad + 4Q^{\mu\nu}\partial_\mu \hat{\hat{Q}}^{(2)}_\nu + 4iB^{\mu\nu}\left[\hat{\hat{Q}}^{(1)}_\mu \stackrel{\star}{,} Q_\nu\right] + 4iQ^{\mu\nu}\left[B_\mu \stackrel{\star}{,} \hat{\hat{Q}}^{(1)}_\nu\right] + \text{irrelevant}, \qquad (4.12)\\
S_{BBQQ_2} &= \int -\frac{1}{2}\left(\partial^\mu \hat{\hat{Q}}^{(1)}_\mu\right)^2 - (\partial_\nu Q^\nu)\left(\partial^\mu \hat{\hat{Q}}^{(2)}_\mu\right) - i(\partial_\nu Q^\nu)\left[B_\mu \stackrel{\star}{,} \hat{\hat{Q}}^{(1)}_\mu\right] \\
&\quad - i\,[B_\mu \stackrel{\star}{,} Q^\mu]\left(\partial^\mu \hat{\hat{Q}}^{(1)}_\mu\right) + \frac{1}{2}\left([B_\mu \stackrel{\star}{,} Q^\mu]\right)^2 + \text{irrelevant}, \qquad (4.13)
\end{aligned}
$$

and

$$
\begin{aligned}
S_{BC\bar{C}} &= -\int \bar{C}\Box\hat{\hat{C}}^{(1)} + i\bar{C}\partial_\mu\,[B^\mu \stackrel{\star}{,} C] + \bar{C}\,[B^\mu \stackrel{\star}{,} \partial_\mu C]\,, \qquad (4.14)\\
S_{BBC\bar{C}} &= \int -i\bar{C}\partial_\mu\left[B^\mu \stackrel{\star}{,} \hat{\hat{C}}^{(1)}\right] + i\,[B^\mu \stackrel{\star}{,} \bar{C}]\,\partial^\mu \hat{\hat{C}}^{(1)} - [B_\mu \stackrel{\star}{,} \bar{C}]\,[B^\mu \stackrel{\star}{,} C] + \text{irrelevant}. \qquad (4.15)
\end{aligned}
$$

Note that we use $B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu$ and $Q_{\mu\nu} \equiv \partial_\mu Q_\nu - \partial_\nu Q_\mu$ in the equations above. Operators $\hat{\hat{Q}}_\mu$ and $\hat{\hat{C}}$ are defined in (D.4) and (D.6), respectively. Here and later "irrelevant" denotes those four-field-interaction terms which do not generate nontrivial nonlocal factor and/or denominator in the tadpole diagrams. Their contributions to tadpole is then zero under dimensional regularization because of the reasons given in the section 3 and the appendix C. They would still be needed for loop corrections to the three and higher point functions. The interaction Feynman rules are read out from the interactions listed above, and given in the appendix E.1.

4.2 One-loop quantum corrections in the background field gauge

The one-loop photon 1-PI two point function computation in the background field gauge consists four diagrams (figures 2–5): the photon self-interacting bubble $B^{\mu\nu}_{\text{BFG}_\text{photon}}$ and tadpole $T^{\mu\nu}_{\text{BFG}_\text{photon}}$, as well as the ghost bubble $B^{\mu\nu}_{\text{BFG}_\text{ghost}}$ and tadpole $T^{\mu\nu}_{\text{BFG}_\text{ghost}}$:

$$
\Gamma^{\mu\nu}_{\text{BFG}} = B^{\mu\nu}_{\text{BFG}_\text{photon}} + T^{\mu\nu}_{\text{BFG}_\text{photon}} + B^{\mu\nu}_{\text{BFG}_\text{ghost}} + T^{\mu\nu}_{\text{BFG}_\text{ghost}}, \qquad (4.16)
$$

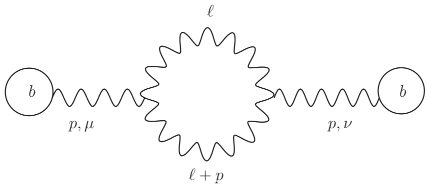

Figure 2. Three-photon bubble contribution to the photon two-point function $B^{\mu\nu}_{\mathrm{BFG_{photon}}}$.

with

$$
\begin{aligned}
B^{\mu\nu}_{\mathrm{BFG_{photon}}} &= \frac{1}{2}\int \frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\rho_1\rho_2}}{\ell^2}\frac{-ig_{\sigma_1\sigma_2}}{(\ell+p)^2}\Gamma^{\mu\rho_1\sigma_2}_{BQQ_{\mathrm{BFG}}}\left(p;\ell,-p-\ell\right)\Gamma^{\nu\rho_2\sigma_2}_{BQQ_{\mathrm{BFG}}}\left(-p;-\ell,p+\ell\right) \\
&= \mathcal{B}^{\mu\nu}_{1_{\mathrm{BFG}}} + \mathcal{B}^{\mu\nu}_{2_{\mathrm{BFG}}}, \qquad (4.17)\\
\mathcal{B}^{\mu\nu}_{1_{\mathrm{BFG}}} &= \frac{1}{2}\int \frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\rho_1\rho_2}}{\ell^2}\frac{-ig_{\sigma_1\sigma_2}}{(\ell+p)^2}\Big(\Gamma^{\mu\rho_1\sigma_2}_{BQQ_{\mathrm{BFG_1}}}\left(p;\ell,-p-\ell\right)\Gamma^{\nu\rho_2\sigma_2}_{BQQ_{\mathrm{BFG_1}}}\left(-p;-\ell,p+\ell\right) \\
&\quad + \Gamma^{\mu\rho_1\sigma_2}_{BQQ_{\mathrm{BFG_1}}}\left(p;\ell,-p-\ell\right)\Gamma^{\nu\rho_2\sigma_2}_{BQQ_{\mathrm{BFG_2}}}\left(-p;-\ell,p+\ell\right) \\
&\quad + \Gamma^{\mu\rho_1\sigma_2}_{BQQ_{\mathrm{BFG_2}}}\left(p;\ell,-p-\ell\right)\Gamma^{\nu\rho_2\sigma_2}_{BQQ_{\mathrm{BFG_1}}}\left(-p;-\ell,p+\ell\right)\Big), \qquad (4.18)\\
\mathcal{B}^{\mu\nu}_{2_{\mathrm{BFG}}} &= \frac{1}{2}\int \frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\rho_1\rho_2}}{\ell^2}\frac{-ig_{\sigma_1\sigma_2}}{(\ell+p)^2}\Gamma^{\mu\rho_1\sigma_2}_{BQQ_{\mathrm{BFG_2}}}\left(p;\ell,-p-\ell\right)\Gamma^{\nu\rho_2\sigma_2}_{BQQ_{\mathrm{BFG_2}}}\left(-p;-\ell,p+\ell\right), \qquad (4.19)
\end{aligned}
$$

$$
\begin{aligned}
T^{\mu\nu}_{\mathrm{BFG_{photon}}} &= \frac{1}{2}\int \frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\rho_1\rho_2}}{\ell^2}\Gamma^{\mu\nu\rho_1\rho_2}_{BBQQ_{\mathrm{BFG}}}\left(p,-p;\ell,-\ell\right) \\
&= \mathcal{T}^{\mu\nu}_{1_{\mathrm{BFG}}} + \mathcal{T}^{\mu\nu}_{2_{\mathrm{BFG}}}, \qquad (4.20)\\
\mathcal{T}^{\mu\nu}_{1_{\mathrm{BFG}}} &= \frac{1}{2}\int \frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\rho_1\rho_2}}{\ell^2}\Gamma^{\mu\nu\rho_1\rho_2}_{BBQQ_{\mathrm{BFG_1}}}\left(p,-p;\ell,-\ell\right), \qquad (4.21)\\
\mathcal{T}^{\mu\nu}_{2_{\mathrm{BFG}}} &= \frac{1}{2}\int \frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\rho_1\rho_2}}{\ell^2}\Gamma^{\mu\nu\rho_1\rho_2}_{BBQQ_{\mathrm{BFG_2}}}\left(p,-p;\ell,-\ell\right), \qquad (4.22)\\
B^{\mu\nu}_{\mathrm{BFG_{ghost}}} &= -\int \frac{d^D\ell}{(2\pi)^D}\frac{i}{\ell^2}\frac{i}{(\ell+p)^2}\Gamma^{\mu}_{Bc\bar{c}_{\mathrm{BFG}}}\left(p;\ell\right)\Gamma^{\nu}_{Bc\bar{c}_{\mathrm{BFG}}}\left(-p;p+\ell\right), \qquad (4.23)\\
T^{\mu\nu}_{\mathrm{BFG_{ghost}}} &= -\int \frac{d^D\ell}{(2\pi)^D}\frac{i}{\ell^2}\Gamma^{\mu\nu}_{BBc\bar{c}_{\mathrm{BFG}}}\left(p,-p;\ell,\ell\right). \qquad (4.24)
\end{aligned}
$$

One can prove that[4]

$$
\mathcal{B}^{\mu\nu}_{2_{\mathrm{BFG}}} + \mathcal{T}^{\mu\nu}_{2_{\mathrm{BFG}}} + B^{\mu\nu}_{\mathrm{BFG_{ghost}}} + T^{\mu\nu}_{\mathrm{BFG_{ghost}}} = 0. \qquad (4.25)
$$

So

$$
\Gamma^{\mu\nu}_{\mathrm{BFG}} = \mathcal{B}^{\mu\nu}_{1_{\mathrm{BFG}}} + \mathcal{T}^{\mu\nu}_{1_{\mathrm{BFG}}}. \qquad (4.26)
$$

[4]This cancellation actually indicates that the stand-alone gauge fixing contribution to the 1PI photon two point function vanishes. Gauge fixing contributions still exist via the products of Γ_1 and Γ_2 in $\mathcal{B}_1$. However if one replaces S_{loop} by S'_{loop}, this effect also disappears because Γ_1 is orthogonal to Γ_2 in that case, the final result for S'_{loop} then goes back to [17] because the background-field splitting becomes trivial in that case.

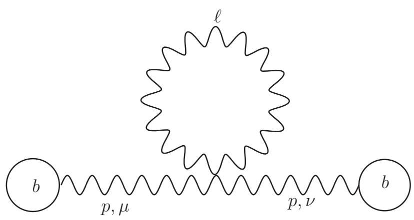

Figure 3. Four-photon tadpole contribution to the photon two-point function $T^{\mu\nu}_{\mathrm{BFG}_{\mathrm{photon}}}$.

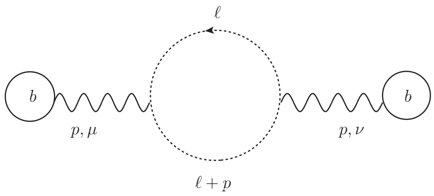

Figure 4. Photon-ghost bubble contribution to the photon two-point function $B^{\mu\nu}_{\mathrm{BFG}_{\mathrm{ghost}}}$.

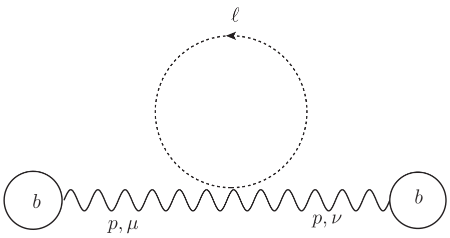

Figure 5. Photon-ghost tadpole contribution to the photon two-point function $T^{\mu\nu}_{\mathrm{BFG}_{\mathrm{ghost}}}$.

Explicit computation then yields

$$\begin{aligned}\mathcal{B}^{\mu\nu}_{1_{\mathrm{BFG}}} = \frac{1}{(4\pi)^2}\Bigg(&\Big(g^{\mu\nu}p^2 - p^\mu p^\nu\Big) \\ &\cdot\left((4\pi\mu^2)^{2-\frac{D}{2}}(p^2)^{\frac{D}{2}-2}2(6-7D)\Gamma\left(1-\frac{D}{2}\right)\mathrm{B}\left(\frac{D}{2},\frac{D}{2}\right)\Big|_{D\to 4-\epsilon}\right. \\ &\left.-12I_{K_0}-16I_{K_1}\right)-g^{\mu\nu}p^2(\theta p)^2T_{-2}-\frac{(\theta p)^\mu(\theta p)^\nu}{(\theta p)^2}\left(\frac{16}{3}T_0+8I^0_K-48p^2I^1_K\right)\Bigg),\end{aligned} \tag{4.27}$$

$$\mathcal{T}^{\mu\nu}_{1_{\mathrm{BFG}}} = \frac{1}{(4\pi)^2}\left(g^{\mu\nu}p^2(\theta p)^2(\theta p)^2T_{-2} - \frac{(\theta p)^\mu(\theta p)^\nu}{(\theta p)^2}\frac{32}{3}T_0\right). \tag{4.28}$$

Thus

$$\Gamma^{\mu\nu}_{\rm BFG} = \frac{1}{(4\pi)^2}\Bigg(\Big(g^{\mu\nu}p^2 - p^\mu p^\nu\Big) \cdot \Bigg((4\pi\mu^2)^{2-\frac{D}{2}}(p^2)^{\frac{D}{2}-2}2(6-7D)\Gamma\left(1-\frac{D}{2}\right){\rm B}\left(\frac{D}{2},\frac{D}{2}\right)\Bigg|_{D\to 4-\epsilon} - 12I_{K_0} - 16I_{K_1}\Bigg) - \frac{(\theta p)^\mu(\theta p)^\nu}{(\theta p)^2}\Big(16T_0 + 8I^0_K - 48p^2 I^1_K\Big)\Bigg). \tag{4.29}$$

This result exactly matches $\hat{\Gamma}^{\mu\nu}_{\rm BFG}$, eqs. (F.16)–(F.18). Using the fact that $T_0 = -2/(\theta p)^2$ [17] one can immediately recover the same quadratic IR divergence equals to $32(\theta p)^\mu(\theta p)^\nu/(\theta p)^4$, which is the same as noncommutative $\mathrm{U}_\star(1)$ theory [6, 7]. Now the UV divergent part of $\Gamma^{\mu\nu}_{\rm BFG}$ at the $D\to 4-\epsilon$ limit reads

$$\Gamma^{\mu\nu}_{\rm BFG}\big|_{\rm UV} = \frac{1}{(4\pi)^2}\Big(g^{\mu\nu}p^2 - p^\mu p^\nu\Big)\frac{22}{3}\left(\frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2)\right). \tag{4.30}$$

This coefficient 22/3 matches exactly the coefficient for $\beta(g)$ of the NC U(1) theory [4, 7].

4.3 One-loop corrections in the noncommutative Feynman gauge

We perform a second test on the gauge-fixing (in-)dependence by shifting from the background field gauge fixing $\hat{D}_\mu[\hat{B}_\mu]\hat{Q}^\mu$ to the NC Feynman gauge fixing (NCFG) $\partial_\mu\hat{Q}^\mu$. The standard background field method procedure then leads us to a modification to the following action

$$\begin{aligned} S^{(1)}_{\mathrm{U(1)_{NCFG}}} = &-\frac{1}{4}\int\Big(\hat{D}_\mu[\hat{B}_\mu]\hat{\hat{Q}}_\nu - \hat{D}_\nu[\hat{B}_\mu]\hat{\hat{Q}}_\mu\Big)^2 - \frac{i}{2}\int\hat{F}^{\mu\nu}[\hat{B}_\mu]\Big[\hat{\hat{Q}}_\mu \stackrel{\star}{,} \hat{\hat{Q}}_\nu\Big] \\ &-\int\left(\frac{1}{2}\Big(\partial_\mu[\hat{B}_\mu]\hat{\hat{Q}}^\mu\Big)^2 + \bar{C}\partial_\mu\hat{D}^\mu[\hat{B}_\mu]\hat{\hat{C}}\right). \end{aligned} \tag{4.31}$$

The resulted Feynman rules are listed in the appendix E.2. In analogy to the background field gauge, we have the following one-loop contributions

$$\Gamma^{\mu\nu}_{\rm NCFG-BFM} = B^{\mu\nu}_{\rm NCFG-BFM_{photon}} + T^{\mu\nu}_{\rm NCFG-BFM_{photon}} + B^{\mu\nu}_{\rm NCFG-BFM_{ghost}} + T^{\mu\nu}_{\rm NCFG-BFM_{ghost}}, \tag{4.32}$$

with

$$\begin{aligned} &B^{\mu\nu}_{\rm NCFG-BFM_{photon}} \\ &= \frac{1}{2}\int\frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\rho_1\rho_2}}{\ell^2}\frac{-ig_{\sigma_1\sigma_2}}{(\ell+p)^2}\Gamma^{\mu\rho_1\sigma_2}_{BQQ_{\rm NCFG-BFM}}(p;\ell,-p-\ell)\,\Gamma^{\nu\rho_2\sigma_2}_{BQQ_{\rm NCFG-BFM}}(-p;-\ell,p+\ell) \\ &= \mathcal{B}^{\mu\nu}_{1_{\rm NCFG-BFM}} + \mathcal{B}^{\mu\nu}_{2_{\rm NCFG-BFM}}, \end{aligned} \tag{4.33}$$

$$\begin{aligned} \mathcal{B}^{\mu\nu}_{1_{\rm NCFG-BFM}} = \frac{1}{2}\int\frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\rho_1\rho_2}}{\ell^2}\frac{-ig_{\sigma_1\sigma_2}}{(\ell+p)^2} &\cdot\Big(\Gamma^{\mu\rho_1\sigma_2}_{BQQ_{\rm NCFG-BFM_1}}(p;\ell,-p-\ell)\,\Gamma^{\nu\rho_2\sigma_2}_{BQQ_{\rm NCFG-BFM_1}}(-p;-\ell,p+\ell) \\ &+\Gamma^{\mu\rho_1\sigma_2}_{BQQ_{\rm NCFG-BFM_1}}(p;\ell,-p-\ell)\,\Gamma^{\nu\rho_2\sigma_2}_{BQQ_{\rm NCFG-BFM_2}}(-p;-\ell,p+\ell) \\ &+\Gamma^{\mu\rho_1\sigma_2}_{BQQ_{\rm NCFG-BFM_2}}(p;\ell,-p-\ell)\,\Gamma^{\nu\rho_2\sigma_2}_{BQQ_{\rm NCFG-BFM_1}}(-p;-\ell,p+\ell)\Big), \end{aligned} \tag{4.34}$$

$$
\begin{aligned}
&\mathcal{B}^{\mu\nu}_{2_{\mathrm{NCFG-BFM}}}\\
&=\frac{1}{2}\int\frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\rho_1\rho_2}}{\ell^2}\frac{-ig_{\sigma_1\sigma_2}}{(\ell+p)^2}\Gamma^{\mu\rho_1\sigma_2}_{BQQ_{\mathrm{NCFG-BFM}_2}}(p;\ell,-p-\ell)\,\Gamma^{\nu\rho_2\sigma_2}_{BQQ_{\mathrm{NCFG-BFM}_2}}(-p;-\ell,p+\ell)\,,
\end{aligned}
\tag{4.35}
$$

$$
\begin{aligned}
T^{\mu\nu}_{\mathrm{NCFG-BFM_{photon}}}&=\frac{1}{2}\int\frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\rho_1\rho_2}}{\ell^2}\Gamma^{\mu\nu\rho_1\rho_2}_{BBQQ_{\mathrm{NCFG-BFM}}}(p,-p;\ell,-\ell)\\
&=\mathcal{T}^{\mu\nu}_{1_{\mathrm{NCFG-BFM}}}+\mathcal{T}^{\mu\nu}_{2_{\mathrm{NCFG-BFM}}},
\end{aligned}
\tag{4.36}
$$

$$
\mathcal{T}^{\mu\nu}_{1_{\mathrm{NCFG-BFM}}}=\frac{1}{2}\int\frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\rho_1\rho_2}}{\ell^2}\Gamma^{\mu\nu\rho_1\rho_2}_{BBQQ_{\mathrm{NCFG-BFM}_1}}(p,-p;\ell,-\ell)\,,
\tag{4.37}
$$

$$
\mathcal{T}^{\mu\nu}_{2_{\mathrm{NCFG-BFM}}}=\frac{1}{2}\int\frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\rho_1\rho_2}}{\ell^2}\Gamma^{\mu\nu\rho_1\rho_2}_{BBQQ_{\mathrm{NCFG-BFM}_2}}(p,-p;\ell,-\ell)\,,
\tag{4.38}
$$

$$
B^{\mu\nu}_{\mathrm{NCFG-BFM_{ghost}}}=-\int\frac{d^D\ell}{(2\pi)^D}\frac{i}{\ell^2}\frac{i}{(\ell+p)^2}\Gamma^{\mu}_{Bc\bar{c}_{\mathrm{NCFG-BFM}}}(p;\ell)\,\Gamma^{\nu}_{Bc\bar{c}_{\mathrm{NCFG-BFM}}}(-p;p+\ell)\,,
\tag{4.39}
$$

$$
T^{\mu\nu}_{\mathrm{NCFG-BFM_{ghost}}}=-\int\frac{d^D\ell}{(2\pi)^D}\frac{i}{\ell^2}\Gamma^{\mu\nu}_{BBc\bar{c}_{\mathrm{NCFG-BFM}}}(p,-p;\ell,\ell)\,.
\tag{4.40}
$$

Again

$$
\mathcal{B}^{\mu\nu}_{2_{\mathrm{NCFG-BFM}}}+\mathcal{T}^{\mu\nu}_{2_{\mathrm{NCFG-BFM}}}+B^{\mu\nu}_{\mathrm{NCFG-BFM_{ghost}}}+T^{\mu\nu}_{\mathrm{NCFG-BFM_{ghost}}}=0,
\tag{4.41}
$$

so

$$
\Gamma^{\mu\nu}_{\mathrm{NCFG-BFM}}=\mathcal{B}^{\mu\nu}_{1_{\mathrm{NCFG-BFM}}}+\mathcal{T}^{\mu\nu}_{1_{\mathrm{NCFG-BFM}}}.
\tag{4.42}
$$

Explicit computation then yields

$$
\begin{aligned}
\mathcal{B}^{\mu\nu}_{1_{\mathrm{NCFG-BFM}}}=\frac{1}{(4\pi)^2}\Bigg(&\Big(g^{\mu\nu}p^2-p^\mu p^\nu\Big)\\
&\cdot\bigg((4\pi\mu^2)^{2-\frac{D}{2}}(p^2)^{\frac{D}{2}-2}2(2-3D)\Gamma\left(1-\frac{D}{2}\right)\mathrm{B}\left(\frac{D}{2},\frac{D}{2}\right)\bigg|_{D\to4-\epsilon}\\
&-8I_{K_0}-16I_{K_1}\bigg)+p^\mu p^\nu(\theta p)^2T_{-2}-\frac{(\theta p)^\mu(\theta p)^\nu}{(\theta p)^2}\left(\frac{16}{3}T_0+8I^0_K-48p^2I^1_K\right)\Bigg),
\end{aligned}
\tag{4.43}
$$

$$
\mathcal{T}^{\mu\nu}_{1_{\mathrm{NCFG-BFM}}}=-\frac{1}{(4\pi)^2}\left(p^\mu p^\nu(\theta p)^2T_{-2}+\frac{(\theta p)^\mu(\theta p)^\nu}{(\theta p)^2}\frac{32}{3}T_0\right).
\tag{4.44}
$$

Consequently

$$
\begin{aligned}
\Gamma^{\mu\nu}_{\mathrm{NCFG-BFM}}=\frac{1}{(4\pi)^2}\Bigg(&\Big(g^{\mu\nu}p^2-p^\mu p^\nu\Big)\\
&\cdot\bigg((4\pi\mu^2)^{2-\frac{D}{2}}(p^2)^{\frac{D}{2}-2}2(2-3D)\Gamma\left(1-\frac{D}{2}\right)\mathrm{B}\left(\frac{D}{2},\frac{D}{2}\right)\bigg|_{D\to4-\epsilon}\\
&-8I_{K_0}-16I_{K_1}\bigg)-\frac{(\theta p)^\mu(\theta p)^\nu}{(\theta p)^2}\left(16T_0+8I^0_K-48p^2I^1_K\right)\Bigg).
\end{aligned}
\tag{4.45}
$$

This result matches the computations in the Feynman gauge without Seiberg-Witten map in the literature [4, 6, 7]. Since the result without Seiberg-Witten map is equivalent to the background field gauge result on shell [19, 24], we conclude that the Seiberg-Witten mapped result here fulfills this equivalence too.

4.4 One-loop corrections in the noncommutative U(1) Super Yang-Mils

We also investigate whether our method can be used to remove non-polynomial UV divergences in the 1-PI two point functions of the superpartners, i.e. the photinos and adjoint scalars. Our starting actions are as follows

$$S_{\text{photino}} = \int i\hat{\bar{\lambda}}\bar{\sigma}^\mu \hat{D}_\mu \hat{\lambda}, \tag{4.46}$$

$$S_{\text{scalar}} = \int \frac{1}{2}\hat{D}_\mu\hat{\phi}\hat{D}^\mu\hat{\phi}. \tag{4.47}$$

In this case after the background-field splitting $\hat{\lambda} = \hat{\lambda}_B + \hat{\lambda}_Q$ and $\hat{\phi} = \hat{\phi}_B + \hat{\phi}_Q$ we must subtract both the equations of motion of superpartner fields, and their contributions as source of the photon equations of motion, the resulted actions for loop computation are listed below

$$\begin{aligned}
S^{(1)}_{\text{photino}} &= \int i\Big(\hat{\bar{\lambda}}_Q\bar{\sigma}^\mu \hat{D}_\mu[\hat{B}_\mu]\hat{\lambda}_Q + i\hat{\bar{\lambda}}_Q\bar{\sigma}^\mu\left[\hat{\hat{Q}}_\mu \overset{\star}{,} \hat{\lambda}_B\right] + i\hat{\bar{\lambda}}_B\bar{\sigma}^\mu\left[\hat{\hat{Q}}_\mu \overset{\star}{,} \hat{\lambda}_Q\right]\Big),\\
S^{(1)}_{\text{scalar}} &= \int \frac{1}{2}\Big(\hat{D}_\mu[\hat{B}_\mu]\hat{\phi}_Q\hat{D}^\mu[\hat{B}_\mu]\hat{\phi}_Q + 2i\left(\hat{D}_\mu[\hat{B}_\mu]\hat{\phi}_B\left[\hat{\hat{Q}}_\mu \overset{\star}{,} \hat{\phi}_Q\right] + \hat{D}_\mu[\hat{B}_\mu]\hat{\phi}_Q\left[\hat{\hat{Q}}_\mu \overset{\star}{,} \hat{\phi}_B\right]\right)\\
&\quad - \left[\hat{\hat{Q}}_\mu \overset{\star}{,} \hat{\phi}_B\right]\left[\hat{\hat{Q}}^\mu \overset{\star}{,} \hat{\phi}_B\right]\Big).
\end{aligned} \tag{4.48}$$

The relevant SW map can be derived using the background-field splitting method in the subsection 2.2 and results [17]. Once we start reading out Feynman rules our first observation is that the superpartner's contribution to the photon effective action is identical to the results in [17]. Therefore we have the same quadratic IR divergence cancellation. The total UV divergence in the background field gauge is now

$$\Gamma^{\mu\nu}_{\text{BFG}-\text{total}}\big|_{\text{UV}} = \frac{1}{(4\pi)^2}\Big(g^{\mu\nu}p^2 - p^\mu p^\nu\Big)\left(\frac{22}{3} - \frac{4}{3}\text{n}_\text{f} - \frac{1}{3}\text{n}_\text{s}\right)\left(\frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2)\right). \tag{4.49}$$

Therefore it vanishes for $\mathcal{N}=4$ SUSY, i.e. when $n_f = 4, n_s = 6$, as expected. The results we have obtained is in full harmony with the results obtained in [27–34] by formulating the theory in terms of noncommutative fields.

We then use the action to derive the Feynman rules for computing the one-loop 1-PI two point functions of the superpartners. The FR results are listed in the appendix E.3. These Feynman rules produce the two diagrams figure 6 and figure 7 for 1-loop photino, as well as two diagrams figure 8 and figure 9 for adjoint scalar two point functions.

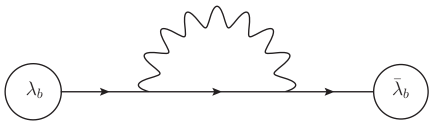

Figure 6. Photino-photon BFM-bubble.

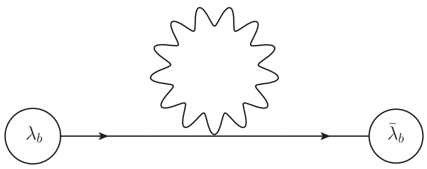

Figure 7. Photino-photon BFM-tadpole.

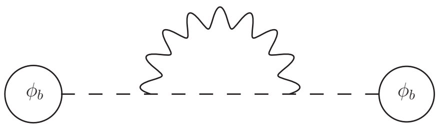

Figure 8. Scalar-photon BFM-bubble.

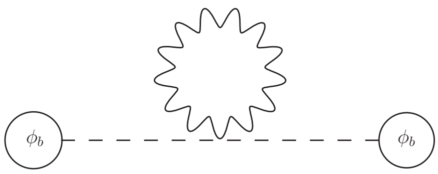

Figure 9. Scalar-photon BFM-tadpole.

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Out of figures 6–9 we read out the following loop integrals for $\mathcal{N}=1$ photino

$$\Sigma^{\dot{\alpha}\alpha}_{\rm BFM} = \Sigma^{\dot{\alpha}\alpha}_{\rm BFM_{bubble}} + \Sigma^{\dot{\alpha}\alpha}_{\rm BFM_{tadpole}}, \tag{4.50}$$

$$\Sigma^{\dot{\alpha}\alpha}_{\rm BFM_{bubble}} = \int \frac{d^D\ell}{(2\pi)^D}\Gamma^{\mu}_{\lambda_B\bar{\lambda}_Q Q}\left(p,\ell{+}p;\ell\right)\frac{i\sigma^\rho(\ell{+}p)_\rho}{(\ell{+}p)^2}\Gamma^{\nu}_{\lambda_Q\bar{\lambda}_B Q}\left(p,\ell{+}p;-\ell\right)\frac{-ig_{\mu\nu}}{\ell^2}, \tag{4.51}$$

$$\Sigma^{\dot{\alpha}\alpha}_{\rm BFM_{tadpole}} = \frac{1}{2}\int \frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\mu\nu}}{\ell^2}\Gamma^{\mu\nu}_{\lambda_B\bar{\lambda}_B QQ}\left(p,p;\ell,-\ell\right), \tag{4.52}$$

and the following for the minimally coupled adjoint scalar

$$\Sigma_{(\phi)_{\rm BFM}} = \Sigma_{(\phi)_{\rm BFM_{bubble}}} + \Sigma_{(\phi)_{\rm BFM_{tadpole}}}, \tag{4.53}$$

$$\Sigma_{(\phi)_{\rm BFM_{bubble}}} = \int \frac{d^D\ell}{(2\pi)^D}\frac{i}{\ell^2}\frac{-ig_{\mu\nu}}{(\ell{+}p)^2}\Gamma^{\mu}_{\phi_B\phi_Q Q}\left(p,\ell;-p-\ell\right)\Gamma^{\nu}_{\phi_B\phi_Q Q}\left(-p,-\ell;p{+}\ell\right), \tag{4.54}$$

$$\Sigma_{(\phi)_{\rm BFM_{tadpole}}} = \frac{1}{2}\int \frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\mu\nu}}{\ell^2}\Gamma^{\mu\nu}_{\phi_B\phi_B QQ}\left(p,-p;\ell,-\ell\right). \tag{4.55}$$

Explicit computation then yields

$$\Sigma^{\dot\alpha\alpha}_{\mathrm{BFM_{bubble}}} = \bar\sigma^\mu p_\mu \frac{1}{(4\pi)^2}\bigg((4\pi\mu^2)^{2-\frac{D}{2}}(p^2)^{\frac{D}{2}-2}(2-D)\Gamma\left(2-\frac{D}{2}\right)\mathrm{B}\left(\frac{D}{2}-1,\frac{D}{2}-1\right)\bigg|_{D\to 4-\epsilon} - (\theta p)^2 T_{-2} - 4I_K^0\bigg), \tag{4.56}$$

$$\Sigma^{\dot\alpha\alpha}_{\mathrm{BFM_{tadple}}} = \bar\sigma^\mu p_\mu \frac{1}{(4\pi)^2}(\theta p)^2 T_{-2}, \tag{4.57}$$

thus

$$\Sigma^{\dot\alpha\alpha}_{\mathrm{BFM}} = \bar\sigma^\mu p_\mu \frac{1}{(4\pi)^2} \cdot\bigg((4\pi\mu^2)^{2-\frac{D}{2}}(p^2)^{\frac{D}{2}-2}(2-D)\Gamma\left(2-\frac{D}{2}\right)\mathrm{B}\left(\frac{D}{2}-1,\frac{D}{2}-1\right)\bigg|_{D\to 4-\epsilon} + 4I_K^0\bigg), \tag{4.58}$$

also

$$\Sigma_{(\phi)\mathrm{BFM_{bubble}}} = p^2\frac{1}{(4\pi)^2} \cdot\bigg(-4(4\pi\mu^2)^{2-\frac{D}{2}}(p^2)^{\frac{D}{2}-2}\Gamma\left(2-\frac{D}{2}\right)\mathrm{B}\left(\frac{D}{2}-1,\frac{D}{2}-1\right)\bigg|_{D\to 4-\epsilon} + T_{-2} + 4T_0 + 2I_K^0\bigg), \tag{4.59}$$

$$\Sigma_{(\phi)\mathrm{BFM_{tadpole}}} = p^2\frac{1}{(4\pi)^2}\left(-T_{-2} + 8T_0\right), \tag{4.60}$$

so

$$\Sigma_{(\phi)\mathrm{BFM}} = p^2\frac{4}{(4\pi)^2} \cdot\bigg(-(4\pi\mu^2)^{2-\frac{D}{2}}(p^2)^{\frac{D}{2}-2}\Gamma\left(2-\frac{D}{2}\right)\mathrm{B}\left(\frac{D}{2}-1,\frac{D}{2}-1\right)\bigg|_{D\to 4-\epsilon} + 3T_0 + 2I_K^0\bigg). \tag{4.61}$$

Comparing (4.58) and (4.61) with their unexpanded counterparts (F.19) and (F.20)–(F.22), one can immediately observe an exact match. On the other hand, this match only occurs when all contributing diagrams are summed together. Individual diagrams, for example (4.59) and (F.21), or (4.60) and (F.22), do not match each other.

Since all other diagrams in the superpartner two point function computation in the SW mapped U(1) NCSYM are identical to the diagrams in the unexpanded theory [17] (see also the short summary in the appendix F.2), we conclude that the full 1-PI two point functions/quadratic part of the background field effective actions are identical up to one-loop in U(1) NCSYM with and without SW map.

5 Discussion and conclusions

We have shown that at the quantum level the θ-exact Seiberg-Witten map provides — at least in perturbative theory with respect to the coupling constant — a dual description, in terms of ordinary fields, of the noncommutative U(N) Yang-Mills theory with or without Supersymmetry. We have shown that by performing appropriate changes of variables in the path integral defining the on-shell DeWitt effective action in dimensional regularization. We have explicitly computed, by using the Feynmann rules derived from the classsical action, the one-loop two-point contribution to the on-shell DeWitt action for U(1) SuperYang-Mills with $\mathcal{N}$=0, 1, 2 and 4 Supersymmetry and found complete agreement with general result obtained by carrying out changes of variables in the path integral. We have also shown that all the nasty non-local noncommutative UV divergences which occur in the one-loop 1PI functional in the Feynman gauge, computed in [10–12, 17], are merely off-shell gauge artifacts since they do not occur in the one-loop two-point contribution to the on-shell DeWitt action — which is a gauge-fixing independent object — and therefore they do not contribute to any physical quantity. We have also shown that the same quadratic noncommutative IR divergences that occur in nonsupersymmetric noncommutative U(N) gauge theories formulated in terms of noncommutative fields occur in the ordinary theory obtained from the former by using the θ-exact Seiberg-Witten map and that this UV/IR mixing effect — signaling a vacuum instability — is a gauge-fixing independent characteristic of the ordinary gauge theory, in keeping with the duality statement. We have also seen that those quadratic noncommutative IR diverges can be removed by considering supersymmetric versions of the theory, a nontrivial effect since supersymmetry is not linearly realized in terms of the ordinary fields [16]. Finally, there remain to be seen how the results presented here carry over to the nonpertubative regime in the coupling constant. In this regard the analysis of the nonperturbative features of $\mathcal{N} = 2$ and 4 supersymmetric gauge theories looks particularly interesting.

Acknowledgments

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A Classical equations of motion for the noncommutative and ordinary fields

In this subsection we prove that the equations of motion are equivalent for the noncommutative and ordinary fields in the NC U(N) gauge theories. We start with the noncommutative fields. The action reads

$$S_{\text{NCYM}} = -\frac{1}{4g^2}\int \text{tr}\left(\hat{F}_{\mu\nu}[\hat{B}_\mu]\hat{F}^{\mu\nu}[\hat{B}_\mu]\right), \tag{A.1}$$

where

$$\hat{F}_{\mu\nu}[\hat{B}_\mu] = \partial_\mu\hat{B}_\nu - \partial_\nu\hat{B}_\mu + i\left[\hat{B}_\mu \stackrel{\star}{,} \hat{B}_\nu\right]. \tag{A.2}$$

If, in terms of the component fields $\hat{B}_\mu = \hat{B}^a_\mu T^a$, than T^a is in the fundamental representation of U(N). The equations of motion for $\hat{B}^a_\mu$ read

$$\text{tr}\left(T^a\hat{D}^\mu[\hat{B}_\mu]\hat{F}_{\mu\nu}[\hat{B}_\mu]\right) = 0, \tag{A.3}$$

which is equivalent to

$$\hat{D}^\mu[\hat{B}_\mu]\hat{F}_{\mu\nu}[\hat{B}_\mu] = 0. \tag{A.4}$$

Now, if B^b_ν and $\hat{B}^a_\mu$ are related by the SW map

$$\begin{aligned}\hat{B}^a_\mu\left[B^b_\nu\right] = B^a_\mu + \sum_{n=2}^{\infty}\int\prod_{i=1}^{n}\frac{d^4p_i}{(2\pi)^4}e^{i\left(\sum\limits_{i=1}^{n}p_i\right)x}\text{tr}\left(T^a\mathfrak{A}^{(n)}_\mu[(a_1,\mu_1,p_1),\dots,(a_n,\mu_n,p_n);\theta]\right)\\ \cdot\tilde{B}^{a_1}_{\mu_1}(p_1)\dots\tilde{B}^{a_n}_{\mu_n}(p_n),\end{aligned} \tag{A.5}$$

and

$$\det\frac{\delta\hat{B}^a_\mu\left[B^b_\nu\right](x)}{B^b_\nu(y)} \neq 0, \tag{A.6}$$

i.e.

$$\begin{aligned}0 = \delta B^a_\mu + \sum_{n=2}^{\infty}\int\prod_{i=1}^{n}\frac{d^4p_i}{(2\pi)^4}e^{i\left(\sum\limits_{i=1}^{n}p_i\right)x}\text{tr}\left(T^a\mathfrak{A}^{(n)}_\mu[(a_1,\mu_1,p_1),\dots,(a_n,\mu_n,p_n);\theta]\right)\\ \cdot n\cdot\tilde{B}^{a_1}_{\mu_1}(p_1)\dots\delta\tilde{B}^{a_n}_{\mu_n}(p_n),\end{aligned} \tag{A.7}$$

has no zero modes (nonzero solutions), i.e. $\hat{B}^a_\mu = \hat{B}^a_\mu\left[B^b_\nu\right]$ can be inverted into $B^a_\mu = B^a_\mu\left[\hat{B}^b_\nu\right]$.

We have that the equation of motion for B^a_μ with action

$$S_{\text{NCYM}} = -\frac{1}{4g^2}\int \text{tr}\left(\hat{F}_{\mu\nu}\left[\hat{B}_\mu[B_\mu]\right]\hat{F}^{\mu\nu}\left[\hat{B}_\mu[B_\mu]\right]\right), \tag{A.8}$$

reads

$$\begin{aligned}0 = \frac{\delta S_{\text{NCYM}}}{\delta B^a_\mu(x)} = \int d^4y\frac{\delta S_{\text{NCYM}}}{\delta\hat{B}^b_\nu(y)}\frac{\delta\hat{B}^b_\nu(y)}{\delta B^a_\mu(x)}\Bigg|_{\hat{B}^a_\mu=\hat{B}^a_\mu[B^b_\nu]} \Longleftrightarrow \frac{\delta S_{\text{NCYM}}}{\delta\hat{B}^a_\mu}\Bigg|_{\hat{B}^a_\mu[B^b_\nu]} = 0\\ \Longleftrightarrow \hat{D}^\mu\left[\hat{B}_\mu[B_\mu]\right]\hat{F}_{\mu\nu}\left[\hat{B}_\mu[B_\mu]\right] = 0.\end{aligned} \tag{A.9}$$

Notice however:

1. For SU(N), SO(N) etc. groups (A.9) is not the equation of motion of B^a_μ since the dependence of S_{NCYM} on B^a_μ is not exhausted by the dependence of $\hat{B}^a_\mu$ on B^a_μ.

2. While the equations of motion of noncommutative and ordinary fields are equivalent, they are not exactly identical. This would affect the subtraction of EOM proportional terms when evaluating the background field effective action and lead to nonidentical off-shell results. As described in the main text, one can obtain exactly identical results in direct computations using noncommutative or ordinary fields only by subtracting the identical EOM proportional terms.

B Some detailed computations

From (2.17) (see also (2.9), (2.13) and (2.14)), one gets

$$\begin{aligned}\frac{\delta\hat{Q}^a_\mu(x)}{\delta Q^b_\nu(y)} &= \frac{1}{\hbar^{\frac{1}{2}}}\,\frac{\delta\hat{A}^a_\mu(x)}{\delta Q^b_\nu(y)}\\ &= \delta^a_b\delta^\nu_\mu\,\delta(x-y)+\sum_{n=2}^{\infty}\int\prod_{i=1}^{n}\frac{d^4p_i}{(2\pi)^4}\Bigg[e^{i\left(\sum\limits_{i=1}^{n}p_i\right)x}\cdot n\,\hbar^{-\frac{1}{2}}\mathrm{tr}\Big(T^a\mathfrak{A}^{(n)}{}_\mu\big[(a_1,\mu_1,p_1),\\ &\qquad\ldots,(a_{n-1},\mu_{n-1},p_{n-1}),(a_n,\mu_n,p_n);\theta\big]\Big)\cdot\tilde{A}^{a_1}_{\mu_1}(p_1)\ldots\tilde{A}^{a_{n-1}}_{\mu_{n-1}}(p_{n-1})\,\frac{\delta\tilde{A}^a_{\mu_n}(p_n)}{\delta Q^b_\nu(y)}\Bigg].\end{aligned}$$

Taking into account (3.4) and using $\tilde{A}^{a_n}_{\mu_n}(p_n)=\tilde{B}^{a_n}_{\mu_n}(p_n)+\hbar^{\frac{1}{2}}\tilde{B}^{a_n}_{\mu_n}(p_n)$ one obtains (3.20) and (3.21).

Let us introduce the following definition

$$\mathcal{M}^{a\,\nu}_{b\,\mu}(x;y)=\sum_{n=2}^{\infty}\int\prod_{i=1}^{n}\frac{d^4p_i}{(2\pi)^4}e^{i\left(\sum\limits_{i=1}^{n-1}p_i\right)x}\,e^{ip_n(x-y)}\mathcal{M}^{(n)}{}^{a\,\nu}_{b\,\mu}(p_1,p_2,\ldots p_{n-1};p_n;\theta),\tag{B.1}$$

where $\mathcal{M}^{(n)}{}^{a\,\nu}_{b\,\mu}(p_1,p_2,\ldots p_{n-1};p_n;\theta)$ has been given in (3.21). Then,

$$\begin{aligned}\ln\,J_1[B,Q] &= \mathrm{Tr}\ln\left(\frac{\delta\hat{Q}^a_\mu(x)}{\delta Q^b_\nu(y)}\right)=\mathrm{Tr}\ln\,\left[\delta^a_b\delta^\nu_\mu\delta(x-y)+\mathcal{M}^{a\,\nu}_{b\,\mu}(x;y)\right]\\ &= \int d^4x\mathcal{M}^{a\,\mu}_{a\,\mu}(x;x)\\ &\quad+\sum_{m=1}^{\infty}\frac{(-1)^m}{m+1}\int d^4x\int\prod_{i=1}^{m}d^4x_i\;\mathcal{M}^{a\,\mu_1}_{a_1\,\mu}(x;x_1)\mathcal{M}^{a_1\,\mu_2}_{a_2\,\mu_1}(x_1;x_2)\cdots\mathcal{M}^{a_m\,\mu}_{a\,\mu_m}(x_m;x).\end{aligned}\tag{B.2}$$

The substitution of (B.1) in the previous equation (B.2) yields

$$\begin{aligned}
\ln\, J_1[B,Q] =& \sum_{n=2}^{\infty}\int d^4x\int\prod_{i=1}^{n}\frac{d^4p_i}{(2\pi)^4}e^{i\left(\sum\limits_{i=1}^{n-1}p_i\right)x}\; e^{ip_n(x-x)}\mathcal{M}^{(n)\,a\,\mu}_{\,a\,\mu}(p_1,p_2,\dots p_{n-1};p_n;\theta)\\
&+\sum_{m=1}^{\infty}\frac{(-1)^m}{m+1}\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\cdots\sum_{n_m=2}^{\infty}\sum_{n_{m+1}=2}^{\infty}\int d^4x\int\prod_{i=1}^{m}d^4x_i\\
&\left\{\left[\int\prod_{i_1=1}^{n_1}\frac{d^4p_{1,i_1}}{(2\pi)^4}e^{i\left(\sum\limits_{i_1=1}^{n_1-1}p_{1,i_1}\right)x}\; e^{ip_{1,n_1}(x-x_1)}\mathcal{M}^{(n_1)\,a\,\mu_1}_{\,a_1\,\mu}(p_{1,1},p_{1,2},\dots,p_{1,n_1-1};p_{1,n_1};\theta)\right]\right.\\
&\cdot\left[\int\prod_{i_2=1}^{n_2}\frac{d^4p_{2,i_2}}{(2\pi)^4}e^{i\left(\sum\limits_{i_2=1}^{n_2-1}p_{2,i_2}\right)x_1}\; e^{ip_{2,n_2}(x_1-x_2)}\mathcal{M}^{(n_2)\,a_1\,\mu_2}_{\,a_2\,\mu_1}(p_{2,1},p_{2,2},\dots,p_{2,n_2-1};p_{2,n_2};\theta)\right]\\
&\cdot\ldots\cdot\\
&\cdot\left[\int\prod_{i_m=1}^{n_m}\frac{d^4p_{m,i_m}}{(2\pi)^4}e^{i\left(\sum\limits_{i_m=1}^{n_m-1}p_{m,i_m}\right)x_{m-1}}\; e^{ip_{m,n_m}(x_{m-1}-x_m)}\right.\\
&\qquad\left.\cdot\,\mathcal{M}^{(n_m)\,a_{m-1}\,\mu_m}_{\,a_m\,\mu_{m-1}}(p_{m,1},p_{m,2},\dots,p_{m,n_m-1};p_{m,n_m};\theta)\right]\\
&\cdot\left[\int\prod_{i_{m+1}=1}^{n_{m+1}}\frac{d^4p_{m+1,i_{m+1}}}{(2\pi)^4}e^{i\left(\sum\limits_{i_{m+1}=1}^{n_{m+1}-1}p_{m+1,i_{m+1}}\right)x_m}\; e^{ip_{m+1,n_{m+1}}(x_m-x)}\right.\\
&\qquad\left.\left.\cdot\,\mathcal{M}^{(n_{m+1})\,a_m\,\mu}_{\phantom{(n_{m+1})}\,a\,\mu_m}(p_{m+1,1},p_{m+1,2},\dots,p_{m+1,n_{m+1}-1};p_{m+1,n_{m+1}};\theta)\right]\right\}.
\end{aligned}\tag{B.3}$$

Introducing the following definitions

$$l_1=\sum_{i_1=1}^{n_1-1}p_{1,i_1},\, l_2=\sum_{i_2=1}^{n_2-1}p_{2,i_2},\dots,l_{m+1}=\sum_{i_{m+1}=1}^{n_{m+1}}p_{m+1,i_{m+1}},$$

and carrying out the integration over x and x_i, $i=1,\dots,m$, one obtains the following product of Dirac deltas

$$\begin{aligned}
&\delta(l_1+p_{1,n_1}-p_{m+1,n_{m+1}})\delta(l_2-p_{1,n_1}+p_{2,n_2})\delta(l_3-p_{2,n_2}+p_{3,n_3})\cdot\cdots\cdot\\
&\cdot\delta(l_m-p_{m-1,n_{m-1}}+p_{m,n_m})\delta(l_{m+1}-p_{m,n_m}+p_{m+1,n_{m+1}}).
\end{aligned}$$

Renaming p_{1,n_1} as q and integrating out p_{2,n_2}, p_{3,n_3}, ... and $p_{m+1,n_{m+1}}$, one removes all Dirac deltas but one, which turns out to be $\delta\left(\sum\limits_{i=1}^{m+1}l_i\right)$, and obtains (3.22).

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C Vanishing integrals in dimensional regularization

In this appendix we shall discuss why the integrals over the internal momentum q that arise in the computation of the Jacobian determinants in sections 3.1 and 3.2 vanish in dimensional regularization. These integrals are of the following type

$$\mathfrak{I} = \int \frac{d^D q}{(2\pi)^D}\, \mathbb{Q}(q)\, \mathbb{I}(q\theta k_i, k_i\theta k_j), \tag{C.1}$$

where $\mathbb{Q}(q) = q^{\rho_1} q^{\rho_2}\cdots q^{\rho_n}$, $q\theta k_i = q_\mu \theta^{\mu\nu} k_{i\nu}$, $i = 1,\ldots,s$, and $k_i\theta k_j = k_{i\mu}\theta^{\mu\nu}k_{j\nu}$, $i,j = 1,\ldots,s$. Here n and s runs over all relevant momenta other than q, in general. The function $\mathbb{I}$ in the integrand of the previous integral is a function of the variables $q\theta k_i$ and $k_i\theta k_j$ *only*.

We shall define the integral (C.1) by Wick rotating the corresponding integral defined for Euclidean signature, a signature which we shall assume for the time being.

The first problem one has to face when defining, in dimensional regularization, the object in (C.1) is the definition of $\theta^{\mu\nu}$ in the infinite dimensional space, E_∞ — see section 4.1 of ref. [35] — of which the momenta q^μ, k_i^μ are elements in dimensional regularization. Let us recall that, to avoid problems with unitarity, our $\theta^{\mu\nu}$ in four dimensions is such that $\theta^{0i} = 0$, $i = 1,2,3$. Hence, by a rotation, this $\theta^{\mu\nu}$ in four dimensions can be transformed into an object whose only non-vanishing components are θ^{23} and θ^{32}. Then, without loss of generality, we shall assume this latter $\theta^{\mu\nu}$ to be our object in four dimensions.

Now, since $\theta^{\mu\nu}$ is an antisymmetric object, its properties depend on the dimension of spacetime. So, as happens with the Levi-Civita tensor and the γ_5 matrix [35], the only consistent way to define it in dimensional regularization is to keep it essentially four-dimensional, since our physical theory is in four dimensions. This amounts to defining $\theta^{\mu\nu}$ in the infinite dimensional space — see section 4.1 of ref. [35] — of dimensional regularization:

$$\begin{aligned} \theta^{\mu\nu} &= \theta, && \text{if}\quad \mu=2,\ \nu=3\,,\\ \theta^{\mu\nu} &= -\theta, && \text{if}\quad \mu=3,\ \nu=2\,,\\ \theta^{\mu\nu} &= 0, && \text{otherwise.} \end{aligned}$$

With this definition of our $\theta^{\mu\nu}$-object in dimensional regularization, one comes to the conclusion that all the vectors $\frac{1}{\theta}\theta^{\mu\nu}k_{i\nu}$, $i = 1,\ldots,n$, belong to the same two-dimensional subspace, E_2, of the infinite dimensional space E_∞. Let us follow ref. [35] and split the vector $q^\mu \in E_\infty$ into two components:

$$q^\mu = q_\perp^\mu + q_\parallel^\mu\,, \tag{C.2}$$

where $q_\parallel^\mu \in E_2$ and $q_\perp^\mu \in E_\perp$, $E_\perp$ being the subspace orthogonal to E_2. Then, using [35], we define the following object in (C.1)

$$\int \frac{d^D q}{(2\pi)^D}\, \mathbb{Q}(q)\, \mathbb{I}(q\theta k_i, k_i\theta k_j) = \frac{1}{(2\pi)^D}\int dl^1 dl^2 \Big\{ \int d^{D-2}q_\perp\, \mathbb{Q}(q)\, \mathbb{I}(q_\parallel\theta k_i, k_i\theta k_j)\Big\}, \tag{C.3}$$

where l^1 and l^2 are the coordinates of $q_\parallel^\mu$ in an orthonormal basis of E_2 and we have taken into account that $q\theta k_i = q_\parallel \theta k_i$.

Now, $\mathbb{I}(q_{\|}\theta k_i, k_i\theta k_j)$ does not depend on $q_{\perp}^{\mu}$, so that

$$\int d^{D-2}q_{\perp}\,\mathbb{Q}(q)\,\mathbb{I}(q_{\|}\theta k_i, k_i\theta k_j) = \mathbb{I}(q_{\|}\theta k_i, k_i\theta k_j)\int d^{D-2}q_{\perp}\,\mathbb{Q}(q). \tag{C.4}$$

But in dimensional regularization tadpole-type integrals — see [35] — vanish:

$$\int d^{D-2}q_{\perp}\,\mathbb{Q}(q) = 0, \tag{C.5}$$

recall that $\mathbb{Q}(q)$ is a monomial. We thus conclude that

$$\int d^{D-2}q_{\perp}\,\mathbb{Q}(q)\,\mathbb{I}(q_{\|}\theta k_i, k_i\theta k_j) = 0, \tag{C.6}$$

so that the right hand side of equation (C.3) vanishes, which in turn implies that

$$\int \frac{d^Dq}{(2\pi)^D}\,\mathbb{Q}(q)\,\mathbb{I}(q\theta k_i, k_i\theta k_j) = 0. \tag{C.7}$$

D Expansion of the action $\hbar^{-1}S_{\mathrm{NCYM}}[\hat{B}_\mu + \hbar^{\frac{1}{2}}\hat{Q}_\mu]$ in terms of $\hbar$

We shall assume that $\hat{D}^\mu\left[\hat{B}_\mu[B_\mu]\right]\hat{F}_{\mu\nu}\left[\hat{B}_\mu[B_\mu]\right] = 0$, then the action is

$$\begin{aligned}
&\frac{1}{\hbar}S_{\mathrm{NCYM}}[\hat{B}_\mu + \hbar^{\frac{1}{2}}\hat{Q}_\mu] \\
&\quad= -\frac{1}{4g^2\hbar}\int \mathrm{tr}\left(\hat{F}_{\mu\nu}[\hat{B}_\mu + \hbar^{\frac{1}{2}}\hat{Q}_\mu]\hat{F}^{\mu\nu}[\hat{B}_\mu + \hbar^{\frac{1}{2}}\hat{Q}_\mu]\right) \\
&\quad= -\frac{1}{4g^2\hbar}\int \left(\hat{F}_{\mu\nu}[\hat{B}_\mu] + \hbar^{\frac{1}{2}}\left(\hat{D}_\mu[\hat{B}_\mu]\hat{Q}_\nu - \hat{D}_\nu[\hat{B}_\mu]\hat{Q}_\mu\right) - \hbar\left[\hat{Q}_\mu \stackrel{\star}{,} \hat{Q}_\nu\right]\right)^2 \\
&\quad= -\frac{1}{4g^2\hbar}\int \mathrm{tr}\left(\hat{F}_{\mu\nu}[\hat{B}_\mu]\hat{F}^{\mu\nu}[\hat{B}_\mu]\right) - \frac{1}{2g^2\hbar^{\frac{1}{2}}}\int \mathrm{tr}\left(\hat{D}^\mu\hat{F}_{\mu\nu}[\hat{B}_\mu]\hat{Q}^\nu\right) \\
&\qquad - \frac{1}{4g^2}\int \mathrm{tr}\left(\hat{D}_\mu[\hat{B}_\mu]\hat{Q}_\nu - \hat{D}_\nu[\hat{B}_\mu]\hat{Q}_\mu\right)^2 + \frac{1}{2g^2}\int \mathrm{tr}\hat{F}^{\mu\nu}[\hat{B}_\mu]\left[\hat{Q}_\mu \stackrel{\star}{,} \hat{Q}_\nu\right] + \mathcal{O}(\hbar^{\frac{1}{2}}).
\end{aligned} \tag{D.1}$$

The second line after the third equality can be neglected because the background field satisfies the equations of motion (A.9) (Kallosh formalism). Therefore

$$\begin{aligned}
S_{\mathrm{NCYM}}[\hat{B}_\mu + \hbar^{\frac{1}{2}}\hat{Q}_\mu] = S_{\mathrm{NCYM}}[\hat{B}_\mu] &- \frac{1}{4g^2}\int \mathrm{tr}\left(\hat{D}_\mu[\hat{B}_\mu]\hat{Q}_\nu - \hat{D}_\nu[\hat{B}_\mu]\hat{Q}_\mu\right)^2 \\
&+ \frac{1}{2g^2}\int \mathrm{tr}\hat{F}^{\mu\nu}[\hat{B}_\mu]\left[\hat{Q}_\mu \stackrel{\star}{,} \hat{Q}_\nu\right] + \mathcal{O}(\hbar^{\frac{1}{2}}).
\end{aligned} \tag{D.2}$$

Now, one extracts the $\mathcal{O}(\hbar^0)$ order terms of $\hat{Q}_\mu$ from (2.17) and deduces that

$$\begin{aligned}
S_{\mathrm{NCYM}}[\hat{B}_\mu + \hbar^{\frac{1}{2}}\hat{Q}_\mu] = S_{\mathrm{NCYM}}[\hat{B}_\mu] &- \frac{1}{4g^2}\int \mathrm{tr}\left(\hat{D}_\mu[\hat{B}_\mu]\hat{\hat{Q}}_\nu - \hat{D}_\nu[\hat{B}_\mu]\hat{\hat{Q}}_\mu\right)^2 \\
&- \frac{i}{2g^2}\int \mathrm{tr}\hat{F}^{\mu\nu}[\hat{B}_\mu]\left[\hat{\hat{Q}}_\mu \stackrel{\star}{,} \hat{\hat{Q}}_\nu\right] + \mathcal{O}(\hbar^{\frac{1}{2}}),
\end{aligned} \tag{D.3}$$

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where

$$\hat{\hat{Q}}_\mu = Q_\mu + \sum_{n=2}^{\infty} \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} e^{i\left(\sum\limits_{i=1}^{n} p_i\right)x} \mathfrak{A}_\mu^{(n)} \left[(a_1, \mu_1, p_1), \ldots, (a_n, \mu_n, p_n); \theta\right] \\ \cdot n \cdot \tilde{B}_{\mu_1}^{a_1}(p_1) \ldots \tilde{B}_{\mu_{n-1}}^{a_{n-1}}(p_{n-1}) \tilde{Q}_{\mu_n}^{a_n}(p_n). \tag{D.4}$$

i.e. $\hat{Q}_\mu = \hat{\hat{Q}}_\mu + \mathcal{O}(\hbar^{\frac{1}{2}})$. Similarly, we can expand the gauge fixing action (2.36) up to the $\hbar^0$ order

$$S_{\rm gf} = \frac{1}{g^2} \int \mathrm{tr} \left(\alpha \hat{F}^2 + \hat{F} \hat{D}_\mu [\hat{B}_\mu] \hat{\hat{Q}}^\mu - \hat{\bar{C}} \hat{D}_\mu [\hat{B}_\mu] \hat{D}^\mu [\hat{B}_\mu] \hat{\hat{C}} \right), \tag{D.5}$$

where

$$\hat{\hat{C}} = \hat{C}\left[C, B_\mu; \theta\right]. \tag{D.6}$$

E Feynman rules in the background field formalism

We list here all Feynman rules the relevant to the computation in section 4. We use the Fourier transformation rule

$$f(x) = \int \frac{d^4 p}{(2\pi)^4} \tilde{f}(p) e^{ipx}, \tag{E.1}$$

and a convention in the vertex diagrams that sets all photon momenta as incoming. The SW map expansion of $\hat{\hat{Q}}_\mu$ for U(1) gauge theory is derived from SW map for unsplitted field

$$\begin{aligned} \hat{A}_\mu = A_\mu &+ \frac{1}{2}\theta^{ij} A_i \star_2 (\partial_j A_\mu + A_{j\mu}) \\ &- \frac{1}{8}\theta^{ij}\theta^{kl} \big[(\partial_i A_\mu + A_{i\mu}) A_k (\partial_l A_j + A_{lj}) - A_i \partial_j (A_k (\partial_l A_\mu + A_{l\mu})) \\ &+ 2A_i (A_{jk} A_{\mu l} - A_k \partial_l A_{j\mu}) \big]_{\star_{3'}} + \mathcal{O}(A^3), \quad A_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \end{aligned} \tag{E.2}$$

the background-field splitting (2.14) and the expansion (D.4). The leading order (in $\hbar$) ghost Seiberg-Witten map $\hat{\hat{C}}$ in U(1) theory is defined as follows

$$\begin{aligned} \hat{\hat{C}} &= \hat{C}\Big[B_\mu, C; \theta\Big] \\ &= C + \frac{1}{2}\theta^{ij} a_i \star_2 \partial_j C \\ &\quad + \frac{1}{8}\theta^{ij}\theta^{kl} \Big[A_i \partial_j (A_k \partial_l C - \partial_i C A_k (\partial_l A_j) + A_{lj} \Big]_{\star_{3'}} + \mathcal{O}\left(A^3\right) C. \end{aligned} \tag{E.3}$$

The generalized star products $\star_2$ and $\star_{3'}$ here and the corresponding nonlocal factors $f_{\star_2}$ and $f_{\star_{3'}}$ below are the same as defined in [17]. Employing all these ingredients we obtain the Feynman rules below for (one) loop computation in the background field formalism in section 4.[5]

In the next two subsections we are giving Feynman rules which generically correspond to the following figures: figure 10, figure 11, figure 12, and figure 13.

[5] Note that there is a sign change in front of the first order SW map expansion terms in (E.2) and (E.3) with respect to [17], which is due to the change of signature in the covariant derivative definition.

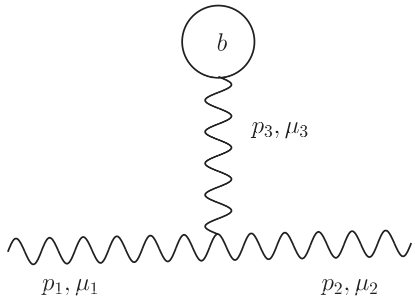

Figure 10. Three-photon BFM FR.

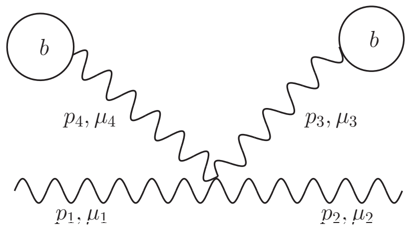

Figure 11. Four-photon BFM FR.

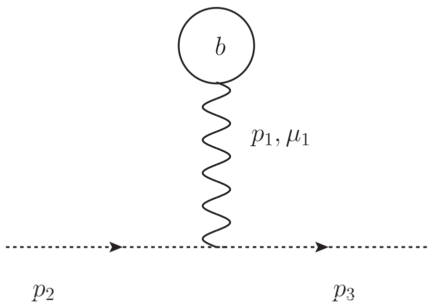

Figure 12. Ghost-photon BFM FR.

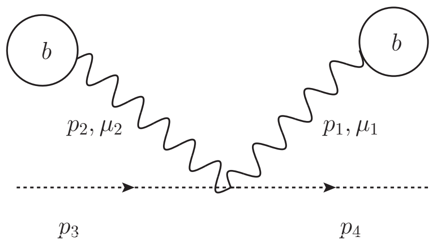

Figure 13. Ghost-2photons BFM FR.

E.1 The background field gauge

$$\Gamma^{\mu\nu_1\nu_2}_{BQQ_{\mathrm{BFG}}}(p;q_1,q_2) = \Gamma^{\mu\nu_1\nu_2}_{BQQ_{\mathrm{BFG}_1}}(p;q_1,q_2) + \Gamma^{\mu\nu_1\nu_2}_{BQQ_{\mathrm{BFG}_2}}(p;q_1,q_2)\,, \tag{E.4}$$

$$\begin{aligned}\Gamma^{\mu\nu_1\nu_2}_{BQQ_{\mathrm{BFG}_1}} = \frac{1}{2}f_{\star_2}(p,q_1)\Bigg(&\Big((p\theta q_1)(P^{\nu_1}g^{\mu\nu_2}-P^{\nu_2}g^{\mu\nu_1})+(p\theta q_2)(P^{\nu_2}g^{\mu\nu_1}-P^{\nu_1}g^{\mu\nu_2})\Big)\\
&-2\Big((\theta p)^{\nu_1}((q_1\cdot p)g^{\mu\nu_2}-q_2^{\mu}p^{\nu_2})+(\theta p)^{\nu_2}((q_1\cdot p)g^{\mu\nu_1}-q_1^{\mu}p^{\nu_1})\Big)\\
&+2\Big((\theta p)^{\mu}((q_1\cdot q_2)g^{\nu_1\nu_2}-q_2^{\nu_1}q_1^{\nu_2})\Big)\\
&+2\Big(q_1^{\mu}(p\theta q_2)g^{\nu_1\nu_2}+q_1^{\mu}(\theta p)^{\nu_2}q_2^{\nu_1}-(q_1\cdot p)(\theta q_2)^{\mu}g^{\nu_1\nu_2}+(q_1\cdot p)q_2^{\nu_1}\theta^{\mu\nu_2}\\
&-q_1^{\nu_2}(p\theta q_2)g^{\mu\nu_1}-(q_1\cdot q_2)(\theta p)^{\nu_2}g^{\mu\nu_1}+q_1^{\nu_2}p^{\nu_1}(\theta q_2)^{\mu}-(q_1\cdot q_2)p^{\nu_1}\theta^{\mu\nu_2}\\
&+q_2^{\mu}(p\theta q_1)g^{\nu_1\nu_2}+q_2^{\mu}(\theta p)^{\nu_1}q_1^{\nu_2}-(q_2\cdot p)(\theta q_1)^{\mu}g^{\nu_1\nu_2}+(q_2\cdot p)q_1^{\nu_2}\theta^{\mu\nu_1}\\
&-q_2^{\nu_1}(p\theta q_1)g^{\mu\nu_2}-(q_1\cdot q_2)(\theta p)^{\nu_1}g^{\mu\nu_2}+q_2^{\nu_1}p^{\nu_2}(\theta q_1)^{\mu}-(q_1\cdot q_2)p^{\nu_2}\theta^{\mu\nu_1}\Big)\Bigg),\end{aligned} \tag{E.5}$$

$$\begin{aligned}\Gamma^{\mu\nu_1\nu_2}_{BQQ_{\mathrm{BFG}_2}} = f_{\star_2}(p,q_1)\Bigg(&((p\theta q_1)g^{\mu\nu_1}q_2^{\nu_2}+(p\theta q_2)g^{\mu\nu_2}q_1^{\nu_1})\\
&+\frac{1}{2}\Big(q_1^{\nu_1}(2q_1^{\nu_2}(\theta q_2)^{\mu}-(q_1\cdot q_2)\theta^{\mu\nu_2}+2q_1^{\mu}(\theta p)^{\nu_2}+(q_1\cdot p)\theta^{\mu\nu_2})\\
&+q_2^{\nu_2}\,(2q_2^{\nu_1}(\theta q_1)^{\mu}-(q_1\cdot q_2)\theta^{\mu\nu_1}+2q_2^{\mu}(\theta p)^{\nu_1}+(q_2\cdot p)\theta^{\mu\nu_1})\Big)\Bigg),\end{aligned} \tag{E.6}$$

$$\Gamma^{\mu_1\mu_2\nu_1\nu_2}_{BBQQ_{\mathrm{BFG}}}(p_1,p_2;q_1,q_2) = \Gamma^{\mu_1\mu_2\nu_1\nu_2}_{BBQQ_{\mathrm{BFG}_1}}(p_1,p_2;q_1,q_2) + \Gamma^{\mu_1\mu_2\nu_1\nu_2}_{BBQQ_{\mathrm{BFG}_2}}(p_1,p_2;q_1,q_2)\,, \tag{E.7}$$

$$\Gamma^{\mu_1\mu_2\nu_1\nu_2}_{BBQQ_{\mathrm{BFG}_1}} = \Gamma^{\mu_1\mu_2\nu_1\nu_2}_{A} + \Gamma^{\mu_1\mu_2\nu_1\nu_2}_{B} + \Gamma^{\mu_1\mu_2\nu_1\nu_2}_{C}, \tag{E.8}$$

$$\Gamma^{\mu_1\mu_2\nu_1\nu_2}_{BBQQ_{\mathrm{BFG}_2}} = \Gamma^{\mu_1\mu_2\nu_1\nu_2}_{BBQQ_{\mathrm{NCFG-BFM}_2}} + \Gamma^{\mu_1\mu_2\nu_1\nu_2}_{F}, \tag{E.9}$$

$$\begin{aligned}\Gamma^{\mu_1\mu_2\nu_1\nu_2}_{A} = {}& V_A^{\mu_1\nu_1\nu_2\mu_2}(p_1,q_1,q_2,p_2)+V_A^{\mu_1\nu_2\nu_1\mu_2}(p_1,q_2,q_1,p_2)\\
&+V_A^{\mu_2\nu_1\nu_2\mu_1}(p_2,q_1,q_2,p_1)+V_A^{\mu_2\nu_2\nu_1\mu_1}(p_2,q_2,q_1,p_1)+V_A^{\nu_1\mu_1\nu_2\mu_2}(q_1,p_1,q_2,p_2)\\
&+V_A^{\nu_1\mu_2\nu_2\mu_1}(q_1,p_2,q_2,p_1)+V_A^{\nu_2\mu_1\nu_1\mu_2}(q_2,p_1,q_1,p_2)+V_A^{\nu_2\mu_2\nu_1\mu_1}(q_2,p_2,q_1,p_1)\\
&+V_A^{\mu_2\nu_1\mu_1\nu_2}(p_2,q_1,p_1,q_2)+V_A^{\mu_2\nu_2\mu_1\nu_1}(p_2,q_2,p_1,q_1)+V_A^{\nu_1\mu_1\mu_2\nu_2}(q_1,p_1,p_2,q_2)\\
&+V_A^{\nu_1\mu_2\mu_1\nu_2}(q_1,p_2,p_1,q_2)+V_A^{\nu_2\mu_1\mu_2\nu_1}(q_2,p_1,p_2,q_1)+V_A^{\nu_2\mu_2\mu_1\nu_1}(q_2,p_2,p_1,q_1)\\
&+V_A^{\mu_1\nu_1\mu_2\nu_2}(p_1,q_1,p_2,q_2)+V_A^{\mu_1\nu_2\mu_2\nu_1}(p_1,q_2,p_2,q_1),\end{aligned} \tag{E.10}$$

$$\begin{aligned}\Gamma^{\mu_1\mu_2\nu_1\nu_2}_{B} = {}& V_B^{\mu_1\nu_1\nu_2\mu_2}(p_1,q_1,q_2,p_2)+V_B^{\mu_1\nu_2\nu_1\mu_2}(p_1,q_2,q_1,p_2)\\
&+V_B^{\nu_1\mu_1\nu_2\mu_2}(q_1,p_1,q_2,p_2)+V_B^{\nu_2\mu_1\nu_1\mu_2}(q_2,p_1,q_1,p_2)+V_B^{\nu_1\mu_1\mu_2\nu_2}(q_1,p_1,p_2,q_2)\\
&+V_B^{\nu_2\mu_1\mu_2\nu_1}(q_2,p_1,p_2,q_1)+V_B^{\mu_1\nu_1\mu_2\nu_2}(p_1,q_1,p_2,q_2)+V_B^{\mu_1\nu_2\mu_2\nu_1}(p_1,q_2,p_2,q_1),\end{aligned} \tag{E.11}$$

$$\Gamma_C^{\mu_1\mu_2\nu_1\nu_2} = V_C^{\nu_1\mu_1\mu_2\nu_2}(q_1,p_1,p_2,q_2) + V_C^{\nu_1\mu_2\mu_1\nu_2}(q_1,p_2,p_1,q_2) \\ + V_C^{\nu_1\mu_1\nu_2\mu_2}(q_1,p_1,q_2,p_2) + V_C^{\nu_1\mu_2\nu_2\mu_1}(k_1,k_2,k_3,k_4) + V_C^{\nu_2\mu_1\mu_2\nu_1}(q_2,p_1,p_2,q_1) \\ + V_C^{\nu_2\mu_2\mu_1\nu_1}(q_2,p_2,p_1,q_1) + V_C^{\nu_2\mu_1\nu_1\mu_2}(q_2,p_1,q_1,p_2) + V_C^{\nu_2\mu_2\nu_1\mu_1}(q_2,p_2,q_1,p_1) \\ + \text{irrelevant}, \tag{E.12}$$

$$\Gamma_F^{\mu_1\mu_2\nu_1\nu_2} = V_F(p_1,q_1,p_2,q_2) + V_F(p_2,q_1,p_1,q_2) + V_F(p_1,q_2,p_2,q_1) + V_F(p_2,q_2,p_1,q_1), \tag{E.13}$$

$$V_A^{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4) = \frac{i}{2} f_{\star_2}(k_1,k_2) f_{\star_2}(k_3,k_4)(k_3\theta k_4) \\ \cdot \Big(2(\theta k_2)^{\mu_1} k_4^{\mu_2} g^{\mu_3\mu_4} - 2(\theta k_2)^{\mu_1} k_4^{\mu_3} g^{\mu_2\mu_4} - (k_2\cdot k_4)\theta^{\mu_1\mu_2} g^{\mu_3\mu_4} + k_2^{\mu_4} k_4^{\mu_3} g^{\mu_1\mu_2}\Big), \tag{E.14}$$

$$V_B^{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4) = -\frac{i}{4} f_{\star_2}(k_1,k_2) f_{\star_2}(k_3,k_4)\Big((k_1\theta k_2)\big((k_3\theta k_4) g^{\mu_1\mu_3} g^{\mu_2\mu_4} \\ + (\theta k_3)^{\mu_4} k_4^{\mu_2} g^{\mu_1\mu_3} - k_3^{\mu_1}(\theta k_4)^{\mu_3} g^{\mu_2\mu_4} + k_3^{\mu_1} k_4^{\mu_2}\big) \\ + (\theta k_1)^{\mu_2}\big(k_2^{\mu_4}(k_3\theta k_4) g^{\mu_1\mu_3} + (k_2\cdot k_4)(\theta k_3)^{\mu_4} g^{\mu_1\mu_3} - k_2^{\mu_4} k_3^{\mu_1}(\theta k_4)^{\mu_3} + (k_2\cdot k_4) k_3^{\mu_1}\theta^{\mu_3\mu_4}\big) \\ - k_1^{\mu_3}\big((\theta k_2)^{\mu_1}(k_3\theta k_4) g^{\mu_2\mu_4} + (\theta k_2)^{\mu_1}(\theta k_3)^{\mu_4} k_4^{\mu_2} - k_2^{\mu_4}(k_3\theta k_4)\theta^{\mu_1\mu_2} - (k_2\cdot k_4)(\theta k_3)^{\mu_4}\theta^{\mu_1\mu_2}\big) \\ + (k_1\cdot k_3)\big((\theta k_2)^{\mu_1}(\theta k_4)^{\mu_3} g^{\mu_2\mu_4} - (\theta k_2)^{\mu_1} k_4^{\mu_2}\theta^{\mu_3\mu_4} - (\theta k_4)^{\mu_3} k_2^{\mu_4}\theta^{\mu_1\mu_2} + (k_2\cdot k_4)\theta^{\mu_1\mu_2}\theta^{\mu_3\mu_4}\big) \\ - (\theta k_2)^{\mu_1}\big(k_2^{\mu_3}(k_3\theta k_4) g^{\mu_2\mu_4} + k_2^{\mu_3}(\theta k_3)^{\mu_4} k_4^{\mu_2} \\ - (k_2\cdot k_3)(\theta k_4)^{\mu_3} g^{\mu_2\mu_4} + (k_2\cdot k_3) k_4^{\mu_2}\theta^{\mu_3\mu_4} - k_2^{\mu_4} k_3^{\mu_2}(\theta k_4)^{\mu_3} - (k_2\cdot k_4) k_3^{\mu_2}\theta^{\mu_3\mu_4}\big) \\ - (\theta k_4)^{\mu_3}\big(k_4^{\mu_1}(k_1\theta k_2) g^{\mu_2\mu_4} + k_4^{\mu_1}(\theta k_1)^{\mu_2} k_2^{\mu_4} \\ - (k_1\cdot k_4)(\theta k_2)^{\mu_1} g^{\mu_2\mu_4} + (k_1\cdot k_4) k_2^{\mu_4}\theta^{\mu_1\mu_1} - k_4^{\mu_2} k_1^{\mu_4}(\theta k_2)^{\mu_1} - (k_1\cdot k_3) k_1^{\mu_4}\theta^{\mu_1\mu_2}\big) \\ + 2(\theta k_2)^{\mu_1}(\theta k_4)^{\mu_3}\big((k_2\cdot k_4) g^{\mu_2\mu_4} - k_2^{\mu_4} k_4^{\mu_2}\big)\Big), \tag{E.15}$$

$$V_C^{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4) = \frac{i}{8} f_{\star_{3'}}(k_2,k_3,k_4)\Big(k_1^2\big(-3(\theta k_3)^{\mu_2} k_4^{\mu_1}\theta^{\mu_3\mu_4} \\ + 4(\theta k_4)^{\mu_2}(\theta k_4)^{\mu_3} g^{\mu_1\mu_4} - k_4^{\mu_1}(\theta k_4)^{\mu_2}\theta^{\mu_3\mu_4} + 2\theta^{\mu_2\mu_3}(k_3\theta k_4) g^{\mu_1\mu_4} + 2(\theta k_3)^{\mu_4} k_4^{\mu_1}\theta^{\mu_2\mu_3} \\ - 2(\theta k_4)^{\mu_3} k_4^{\mu_1}\theta^{\mu_2\mu_4} + 4(\theta k_2)^{\mu_4}(\theta k_4)^{\mu_3} g^{\mu_1\mu_2} - 2(k_2\theta k_4) g^{\mu_1\mu_2}\theta^{\mu_3\mu_4} \\ - 2k_2^{\mu_1}(\theta k_4)^{\mu_3}\theta^{\mu_2\mu_4} - k_2^{\mu_1}(\theta k_4)^{\mu_2}\theta^{\mu_3\mu_4}\big) - k_1^{\mu_1}\big(-3(k_1\cdot k_4)(\theta k_3)^{\mu_2}\theta^{\mu_3\mu_4} \\ + 4k_1^{\mu_4}(\theta k_4)^{\mu_2}(\theta k_4)^{\mu_3} - (k_1\cdot k_4)(\theta k_4)^{\mu_2}\theta^{\mu_3\mu_4} - 2k_1^{\mu_4}(k_3\theta k_4)\theta^{\mu_2\mu_3} \\ - 2(k_1\cdot k_4)(\theta k_3)^{\mu_4}\theta^{\mu_2\mu_3} - 2(k_1\cdot k_4)(\theta k_4)^{\mu_3}\theta^{\mu_2\mu_4} + 4k_1^{\mu_2}(\theta k_2)^{\mu_4}(\theta k_4)^{\mu_3} \\ + 2k_1^{\mu_2}(k_2\theta k_4)\theta^{\mu_3\mu_4} + 2(k_1\cdot k_2)(\theta k_4)^{\mu_3}\theta^{\mu_2\mu_4} - (k_1\cdot k_2)(\theta k_4)^{\mu_2}\theta^{\mu_3\mu_4}\big)\Big), \tag{E.16}$$

$$V_F^{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4) = -\frac{i}{2} f_{\star_2}(k_1,k_2) f_{\star_2}(k_3,k_4)\cdot\Big((k_3\theta k_4) k_4^{\mu_4}\big(2(\theta k_2)^{\mu_1} g^{\mu_2\mu_3} \\ - k_2^{\mu_3}\theta^{\mu_1\mu_2} + 2(\theta k_1)^{\mu_2} g^{\mu_1\mu_3} + \theta^{\mu_1\mu_2} k_1^{\mu_3}\big) - (k_1\theta k_2) g^{\mu_1\mu_2}\big(2(k_3+k_4)^{\mu_4}(\theta k_4)^{\mu_3} \\ - (k_3+k_4)\cdot k_4\theta^{\mu_3\mu_4} + 2(k_3+k_4)^{\mu_3}(\theta k_3)^{\mu_4} + k_3\cdot(k_3+k_4)\theta^{\mu_3\mu_4}\big) \\ + (k_1\theta k_2)(k_3\theta k_4) g^{\mu_1\mu_2} g^{\mu_3\mu_4}\Big), \tag{E.17}$$

$$\Gamma^\mu_{Bc\bar{c}_{\text{BFG}}}(p;q) = f_{\star_2}(p,q)\left(-\frac{1}{2}(p+q)^2(\theta q)^\mu + (p\theta q)(p+2q)^\mu\right), \tag{E.18}$$

$$\Gamma^{\mu_1\mu_2}_{BBc\bar{c}_{\rm BFG}}\left(p_1,p_2;q,q'\right) = \Gamma^{\mu\nu_1\nu_2}_{BBc\bar{c}_{\rm NCFG-BFM}}\left(p_1,p_2;q,q'\right) - if_{\star_2}(p_1,q')f_{\star_2}(p_2,q)(p_1\theta q')\left(-\frac{1}{2}(\theta q)^{\mu_2}(p_2+q)^{\mu_1}+g^{\mu_1\mu_2}(p_2\theta q)\right) - if_{\star_2}(p_2,q')f_{\star_2}(p_1,q)(p_2\theta q')\left(-\frac{1}{2}(\theta q)^{\mu_2}(p_1+q)^{\mu_1}+g^{\mu_1\mu_2}(p_1\theta q)\right) + \text{irrelevant}. \tag{E.19}$$

E.2 The noncommutative Feynman gauge

$$\Gamma^{\mu\nu_1\nu_2}_{BQQ_{\rm NCFG-BFM}}\left(p;q_1,q_2\right) = \Gamma^{\mu\nu_1\nu_2}_{BQQ_{\rm NCFG-BFM_1}}\left(p;q_1,q_2\right) + \Gamma^{\mu\nu_1\nu_2}_{BQQ_{\rm NCFG-BFM_2}}\left(p;q_1,q_2\right), \tag{E.20}$$

$$\Gamma^{\mu\nu_1\nu_2}_{BQQ_{\rm NCFG-BFM_1}} = \Gamma^{\mu\nu_1\nu_2}_{BQQ_{\rm BFG_1}}, \tag{E.21}$$

$$\Gamma^{\mu\nu_1\nu_2}_{BQQ_{\rm NCFG-BFM_2}} = \frac{1}{2}f_{\star_2}(p,q_1)\Big(q_1^{\nu_1}(2q_1^{\nu_2}(\theta q_2)^{\mu}-(q_1\cdot q_2)\theta^{\mu\nu_2}+2q_1^{\mu}(\theta p)^{\nu_2}+(q_1\cdot p)\theta^{\mu\nu_2}) + q_2^{\nu_2}\left(2q_2^{\nu_1}(\theta q_1)^{\mu}-(q_1\cdot q_2)\theta^{\mu\nu_1}+2q_2^{\mu}(\theta p)^{\nu_1}+(q_2\cdot p)\theta^{\mu\nu_1}\right)\Big), \tag{E.22}$$

$$\Gamma^{\mu_1\mu_2\nu_1\nu_2}_{BBQQ_{\rm NCFG-BFM}} = \Gamma^{\mu_1\mu_2\nu_1\nu_2}_{BBQQ_{\rm NCFG-BFM_1}} + \Gamma^{\mu_1\mu_2\nu_1\nu_2}_{BBQQ_{\rm NCFG-BFM_2}}, \tag{E.23}$$

$$\Gamma^{\mu_1\mu_2\nu_1\nu_2}_{BBQQ_{\rm NCFG-BFM_1}} = \Gamma^{\mu_1\mu_2\nu_1\nu_2}_{BBQQ_{\rm BFG_1}}, \tag{E.24}$$

$$\Gamma^{\mu_1\mu_2\nu_1\nu_2}_{BBQQ_{\rm NCFG-BFM_2}} = \Gamma^{\mu_1\mu_2\nu_1\nu_2}_{D} + \Gamma^{\mu_1\mu_2\nu_1\nu_2}_{E}, \tag{E.25}$$

$$\Gamma^{\mu_1\mu_2\nu_1\nu_2}_{D} = V_D^{\nu_1\mu_1\mu_2\nu_2}(q_1,p_1,p_2,q_2) + V_D^{\nu_1\mu_2\mu_1\nu_2}(q_1,p_2,p_1,q_2) + V_D^{\nu_1\mu_1\nu_2\mu_2}(q_1,p_1,q_2,p_2) + V_D^{\nu_1\mu_2\nu_2\mu_1}(k_1,k_2,k_3,k_4) + V_D^{\nu_2\mu_1\mu_2\nu_1}(q_2,p_1,p_2,q_1) + V_D^{\nu_2\mu_2\mu_1\nu_1}(q_2,p_2,p_1,q_1) + V_D^{\nu_2\mu_1\nu_1\mu_2}(q_2,p_1,q_1,p_2) + V_D^{\nu_2\mu_2\nu_1\mu_1}(q_2,p_2,q_1,p_1) + \text{irrelevant}, \tag{E.26}$$

$$\Gamma^{\mu_1\mu_2\nu_1\nu_2}_{E} = V_E(p_1,q_1,p_2,q_2) + V_E(p_2,q_1,p_1,q_2) + V_E(p_1,q_2,p_2,q_1) + V_E(p_2,q_2,p_1,q_1), \tag{E.27}$$

$$V_D^{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4) = \frac{i}{8}f_{\star_{3'}}(k_2,k_3,k_4)k_1^{\mu_1}\big(-3(k_1\cdot k_4)(\theta k_3)^{\mu_2}\theta^{\mu_3\mu_4} + 4k_1^{\mu_4}(\theta k_4)^{\mu_2}(\theta k_4)^{\mu_3} - (k_1\cdot k_4)(\theta k_4)^{\mu_2}\theta^{\mu_3\mu_4} - 2k_1^{\mu_4}(k_3\theta k_4)\theta^{\mu_2\mu_3} - 2(k_1\cdot k_4)(\theta k_3)^{\mu_4}\theta^{\mu_2\mu_3} - 2(k_1\cdot k_4)(\theta k_4)^{\mu_3}\theta^{\mu_2\mu_4} + 4k_1^{\mu_2}(\theta k_2)^{\mu_4}(\theta k_4)^{\mu_3} + 2k_1^{\mu_2}(k_2\theta k_4)\theta^{\mu_3\mu_4} + 2(k_1\cdot k_2)(\theta k_4)^{\mu_3}\theta^{\mu_2\mu_4} - (k_1\cdot k_2)(\theta k_4)^{\mu_2}\theta^{\mu_3\mu_4}\big), \tag{E.28}$$

$$V_E^{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4) = -\frac{i}{8}f_{\star_2}(k_1,k_2)f_{\star_2}(k_3,k_4) \cdot((k_1+k_2)^{\mu_2}(\theta k_2)^{\mu_1}-(k_1+k_2)\cdot k_2\theta^{\mu_1\mu_2})((k_3+k_4)^{\mu_4}(\theta k_4)^{\mu_3}-(k_3+k_4)\cdot k_4\theta^{\mu_3\mu_4}), \tag{E.29}$$

$$\Gamma^{\mu}_{Bc\bar{c}_{\rm NCFG-BFM}}\left(p;q\right) = f_{\star_2}(p,q)\left(-\frac{1}{2}(p+q)^2(\theta q)^{\mu}+(p\theta q)(p+q)^{\mu}\right) \tag{E.30}$$

$$\Gamma^{\mu_1\mu_2}_{BBc\bar{c}_{\rm NCFG-BFM}}\left(p_1,p_2;q,q'\right) = \frac{i}{2}\Big(f_{\star_2}(p_1,q')f_{\star_2}(p_2,q)q'^{\mu_1}(p_1\theta q')(\theta q)^{\mu_2} + f_{\star_2}(p_2,q')f_{\star_2}(p_1,q)q'^{\mu_2}(p_2\theta q')(\theta q)^{\mu_1}\Big) + \text{irrelevant}. \tag{E.31}$$

E.3 Feynman rules for the noncommutative U(1) Super Yang-Mils

We list here only the couplings involving background photino and antiphotino fields λ_B and $\bar\lambda_B$ as well as background adjoint scalar field(s) ϕ_B, since the coupling between background photon field B_μ and quantum fluctuations of photino, antiphotino and adjoint scalar(s) are identical to those in [17]. Figures corresponding to the Feynman rules in this subsection are: figure 14, figure 15, figure 16, figure 17, and figure 18.

$$\Gamma^{\mu}_{\lambda_B\bar\lambda_Q Q}(p,k;q) = i f_{\star_2}(p,q)\left(\gamma^\mu(p\theta q)-(\theta q)^\mu(\not p+\not q)\right), \tag{E.32}$$

$$\Gamma^{\mu}_{\lambda_Q\bar\lambda_B Q}(p,k;q) = i f_{\star_2}(p,q)\left(\gamma^\mu(p\theta k)+(\theta p+\theta k)^\mu\not q\right), \tag{E.33}$$

$$\begin{aligned}\Gamma^{\mu_1\mu_2}_{\lambda_B\bar\lambda_B QQ}(p,p';q_1,q_2) = {}& i f_{\star_2}(p',q_1) f_{\star_2}(p,q_2)\Big(-(\not q_2+\not p)(\theta p')^{\mu_1}(\theta p)^{\mu_2}\\&+(q_1\theta(q_2+p))(\theta p)^{\mu_2}\bar\sigma^{\mu_1}-(\theta p')^{\mu_1}(q_2\theta p_2)\bar\sigma^{\mu_2}\Big)\\&+i f_{\star_2}(p',q_2) f_{\star_2}(p,q_1)\Big(-(\not q_1+\not p)(\theta p')^{\mu_2}(\theta p)^{\mu_1}\\&+(q_2\theta(q_1+p))(\theta p)^{\mu_1}\bar\sigma^{\mu_2}-(\theta p')^{\mu_2}(q_1\theta p)\bar\sigma^{\mu_1}\Big),\end{aligned} \tag{E.34}$$

$$\Gamma^{\mu}_{\phi_B\phi_Q Q}(p,k;q) = -f_{\star_2}(p,q)(k^2(\theta p)^\mu+p^\mu(q\theta k)+k^\mu(p\theta k)), \tag{E.35}$$

$$\begin{aligned}&\Gamma^{\mu_1\mu_2}_{\phi_B\phi_B QQ}(p_1,p_2;q_1,q_2)\\&=\frac{i}{2}f_{\star_2}(p_1,q_1)f_{\star_2}(p_2,q_2)\Big((q_1\theta p_1)(q_2\theta p_2)g^{\mu_1\mu_2}+(p_1+q_1)\cdot(p_2+q_2)(\theta p_1)^{\mu_1}(\theta p_2)^{\mu_2}\\&\quad+2(\theta p_1)^{\mu_1}(q_2\theta p_2)p_2^{\mu_2}-2(p_1+q_1)^{\mu_2}(\theta p_1)^{\mu_1}(q_2\theta p_2)\Big)\\&\quad+\frac{i}{2}f_{\star_2}(p_2,q_1)f_{\star_2}(p_1,q_2)\Big((q_1\theta p_2)(q_2\theta p_1)g^{\mu_1\mu_2}+(p_2+q_1)\cdot(p_1+q_2)(\theta p_2)^{\mu_1}(\theta p_1)^{\mu_2}\\&\quad+2(\theta p_2)^{\mu_1}(q_2\theta p_1)p_1^{\mu_2}-2(p_2+q_1)^{\mu_2}(\theta p_2)^{\mu_1}(q_2\theta p_1)\Big)\\&\quad+\frac{i}{2}f_{\star_2}(p_1,q_2)f_{\star_2}(p_2,q_1)\Big((q_2\theta p_1)(q_1\theta p_2)g^{\mu_1\mu_2}+(p_1+q_2)\cdot(p_2+q_1)(\theta p_1)^{\mu_2}(\theta p_2)^{\mu_1}\\&\quad+2(\theta p_1)^{\mu_2}(q_1\theta p_2)p_2^{\mu_1}-2(p_1+q_2)^{\mu_1}(\theta p_1)^{\mu_2}(q_1\theta p_2)\Big)\\&\quad+\frac{i}{2}f_{\star_2}(p_1,q_1)f_{\star_2}(p_2,q_2)\Big((q_1\theta p_1)(q_2\theta p_2)g^{\mu_1\mu_2}+(p_1+q_1)\cdot(p_2+q_2)(\theta p_1)^{\mu_1}(\theta p_2)^{\mu_2}\\&\quad+2(\theta p_1)^{\mu_1}(q_1\theta p_1)p_2^{\mu_2}-2(p_2+q_2)^{\mu_1}(\theta p_2)^{\mu_2}(q_1\theta p_1)\Big).\end{aligned} \tag{E.36}$$

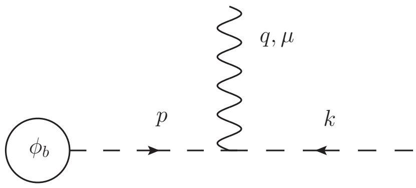

Figure 14. Scalar-photon BFM FR.

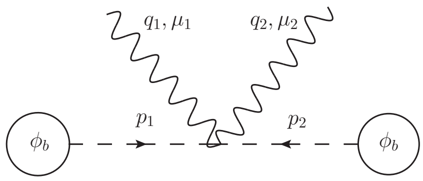

Figure 15. Scalar-2photons BFM FR.

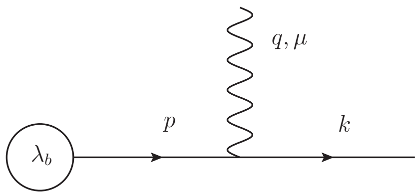

Figure 16. Fermion-photon BFM FR.

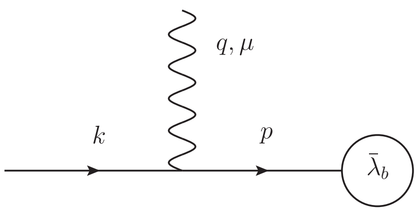

Figure 17. Antifermion-photon BFM FR.

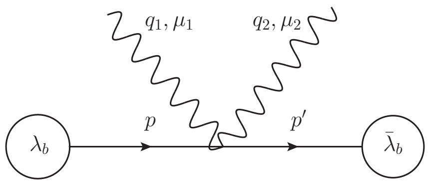

Figure 18. Fermions-2photons BFM FR.

F Evaluation of DeWitt effective action in terms of noncommutative fields

We accumulate the reference results for section 4, i.e. the one-loop quantum corrections to the quadratic part of the DeWitt effective action of the NC U(1) Super Yang Mills. We first give the model setting, then the results of relevant one-loop diagrams.

F.1 The noncommutative Yang-Mils theory

Let's first handle the U(N) NCYM only, then extend the results to its supersymmetrization. The NCYM action is the usual one

$$S_{\rm NCYM}=-\frac{1}{4g^2}\int \mathrm{tr}\Big(\hat F_{\mu\nu}\hat F^{\mu\nu}\Big), \tag{F.1}$$

$$\hat F_{\mu\nu}=\partial_\mu\hat A_\nu-\partial_\nu\hat A_\mu+i\Big[\hat A_\mu \stackrel{\star}{,} \hat A_\nu\Big],\quad \hat A_\mu=\hat A^a_\mu T^a, \tag{F.2}$$

$$\hat\delta_{\rm BRS}\hat A_\mu=\hat D_\mu\hat C=\partial_\mu\hat C+i\Big[\hat A_\mu \stackrel{\star}{,} \hat C\Big],\qquad \hat C=\hat C^a T^a. \tag{F.3}$$

Background field quantization follows the BRST procedure below:

$$\hat A_\mu\Longrightarrow \hat B_\mu+\hbar^{\frac12}\hat Q_\mu,\ \ \hat Q_\mu=\hat Q^a_\mu T^a, \tag{F.4}$$

$$\hat\delta_{\rm BRS}\hat B_\mu=0,\ \ \hbar\hat\delta_{\rm BRS}\hat Q_\mu=\hat D_\mu\big[\hat B_\mu+\hbar^{\frac12}\hat Q_\mu\big]\hat C=\partial_\mu\hat C+i\Big[\hat B_\mu+\hbar^{\frac12}\hat Q_\mu \stackrel{\star}{,} \hat C\Big]. \tag{F.5}$$

Next we introduce the DeWitt effective action $\hat\Gamma_{\rm DeW}\big[\hat B_\mu\big]$ in the background field gauge

$$e^{\frac{i}{\hbar}\hat\Gamma_{\rm BFG}\left[\hat B_\mu\right]}=\int d\hat Q^a_\mu d\hat C^a d\bar C^a dF^a e^{\frac{i}{\hbar}S_{\rm NCYM}\left[\hat B_\mu+\hbar^{\frac12}\hat Q_\mu\right]+iS_{\rm BFG}\left[\hat B_\mu,\hat Q_\mu\right]}, \tag{F.6}$$

with

$$S_{\rm NCYM}\big[\hat B_\mu+\hbar^{\frac12}\hat Q_\mu\big]=-\frac{1}{4g^2}\int \mathrm{tr}\left(\hat F_{\mu\nu}\big[\hat B_\mu+\hbar^{\frac12}\hat Q_\mu\big]\hat F^{\mu\nu}\big[\hat B_\mu+\hbar^{\frac12}\hat Q_\mu\big]\right), \tag{F.7}$$

$$S_{\rm BFG}\big[\hat B_\mu,\hat Q_\mu\big]=\frac{\hbar}{g^2}\int \mathrm{tr}\hat\delta_{\rm BRS}\hat{\bar C}\left(\alpha F+\hat D_\mu\big[\hat B_\mu\big]\hat Q^\mu\right), \tag{F.8}$$

and

$$\hat\delta_{\rm BRS}\bar C=\hbar^{-\frac12}\hat F,\ \ \hat\delta_{\rm BRS}F=0. \tag{F.9}$$

The one-loop contribution $\hat\Gamma^{(1)}_{\rm BFG}\big[\hat B_\mu\big]$ to $\hat\Gamma_{\rm DeW}\big[\hat B_\mu\big]$ corresponds to the $\hbar$ order expansion of the latter

$$\hat\Gamma_{\rm BFG}\big[\hat B_\mu\big]=\hat\Gamma^{(0)}_{\rm BFG}\big[\hat B_\mu\big]+\hbar\hat\Gamma^{(1)}_{\rm BFG}\big[\hat B_\mu\big]+\dots \tag{F.10}$$

To evaluate it we first expand the corresponding classical actions to the appropriate order

$$\begin{aligned}\hbar^{-1}S_{\rm NCYM}\big[\hat B_\mu+\hbar^{\frac12}\hat Q_\mu\big]=&-\frac{1}{4g^2\hbar}\int \mathrm{tr}\left(\hat F_{\mu\nu}\big[\hat B_\mu\big]\hat F^{\mu\nu}\big[\hat B_\mu\big]\right)\\&-\frac{1}{2g^2\hbar^{\frac12}}\int \mathrm{tr}\left(\hat D^\mu\hat F_{\mu\nu}\big[\hat B_\mu\big]\hat Q^\nu\right)\\&-\frac{1}{4g^2}\int \mathrm{tr}\left(\hat D_\mu\big[\hat B_\mu\big]\hat Q_\nu-\hat D_\nu\big[\hat B_\mu\big]\hat Q_\mu\right)^2\\&+\frac{1}{2g^2}\int \mathrm{tr}\hat F^{\mu\nu}\big[\hat B_\mu\big]\Big[\hat Q_\mu \stackrel{\star}{,} \hat Q_\nu\Big]+\mathcal O(\hbar^{\frac12}),\end{aligned} \tag{F.11}$$

$$S_{\rm BFG} = \int {\rm tr}\left(\alpha\hat F^2 + \hat F\hat D_\mu[\hat B_\mu]\hat Q^\mu - \bar C\hat D_\mu[\hat B_\mu]\hat D^\mu[\hat B_\mu]\hat C + \mathcal{O}(\hbar^{\frac{1}{2}})\right). \tag{F.12}$$

Now, let's choose $\hat B_\mu$ on-shell, i.e. $\hat D^\mu[\hat B_\mu]\hat F_{\mu\nu}[\hat B_\mu] = 0$. Then, substituting (F.10), (F.11) and (F.12) in (F.6), one gets

$$\Gamma^{(0)}_{\rm BFG}[\hat B_\mu] = S_{\rm NCYM}[\hat B_\mu], \tag{F.13}$$

$$\hat\Gamma^{(1)}_{\rm BFG}[\hat B_\mu] = -i\ln\int d\hat Q^a_\mu d\hat C^a d\bar C^a dF^a e^{\frac{i}{\hbar}S^{(1)}_{\rm NC}}, \tag{F.14}$$

with

$$\begin{aligned} S^{(1)}_{\rm NC} = &-\frac{1}{4g^2}\int {\rm tr}\left(\hat D_\mu[\hat B_\mu]\hat Q_\nu - \hat D_\nu[\hat B_\mu]\hat Q_\mu\right)^2 + \frac{1}{2g^2}\int {\rm tr}\hat F^{\mu\nu}[\hat B_\mu]\left[\hat Q_\mu \stackrel{\star}{,} \hat Q_\nu\right] \\ &+\int {\rm tr}\left(\alpha\hat F^2 + \hat F\hat D_\mu[\hat B_\mu]\hat Q^\mu - \bar C\hat D_\mu[\hat B_\mu]\hat D^\mu[\hat B_\mu]\hat C\right). \end{aligned} \tag{F.15}$$

Restrict (F.15) to U(1) and $\alpha = 1$, the 1-loop 1PI photon two point function is then evaluated as the sum over 1-loop 1PI diagrams with all $\hat B_\mu$ external lines. There are four diagrams in total, which can be separated into two parts: the bubble part which sums over the photon and ghost bubble diagrams and tadpole part which sums over the photon and ghost tadpole diagrams. Consequently the final result is as follows

$$\hat\Gamma^{\mu\nu}_{\rm BFG} = \hat B^{\mu\nu}_{\rm BFG} + \hat T^{\mu\nu}_{\rm BFG}, \tag{F.16}$$

$$\begin{aligned} \hat B^{\mu\nu}_{\rm BFG} = \frac{1}{(4\pi)^2}\Bigg(&\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right) \\ &\cdot\left((4\pi\mu^2)^{2-\frac{D}{2}}(p^2)^{\frac{D}{2}-2}2(6-7D)\Gamma\left(1-\frac{D}{2}\right){\rm B}\left(\frac{D}{2},\frac{D}{2}\right)\Bigg|_{D\to 4-\epsilon}\right. \\ &\left. - 12I_{K_0} - 16I_{K_1}\right) - \frac{(\theta p)^\mu(\theta p)^\nu}{(\theta p)^2}\left(16T_0 + 8I^0_K - 48p^2I^1_K\right) + g_{\mu\nu}8T_0\Bigg), \end{aligned} \tag{F.17}$$

$$\hat T^{\mu\nu}_{\rm BFG} = -\frac{1}{(4\pi)^2}g_{\mu\nu}8T_0. \tag{F.18}$$

F.2 The U(1) noncommutative Super Yang-Mils theory

Now we shift to the supersymmetrization of the U(1) theory. As discussed in [17], this sector contains the photino(s) for $\mathcal{N} = 1, 2, 4$ and adjoint scalars for $\mathcal{N} = 2, 4$. The interaction between photinos and adjoint scalars remain the same before and after SW map [17], therefore we are not going to repeat them here. Using (4.48) without SW map we obtain one self-energy/bubble diagram figure 19 for photino, as well as a bubble diagram figure 20 and a tadpole diagram figure 21 for adjoint scalar.

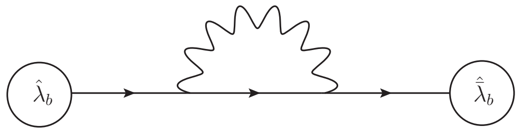

Figure 19. $\mathcal{N}$=1 photino-photon bubble: $\Sigma^{\dot\alpha\alpha}_{(\hat\lambda_b)}(p)_{\rm bub}$.

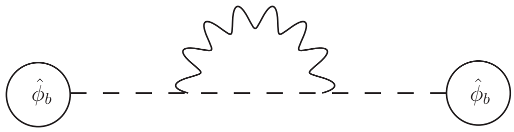

Figure 20. $\mathcal{N}$=2 scalar-photon bubble: $\Sigma_{(\hat\phi_b)}(p)_{\rm bub}$.

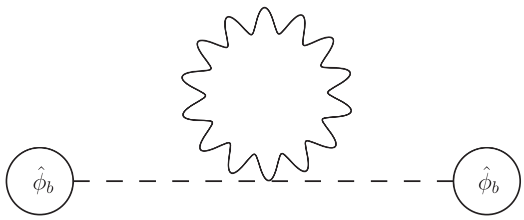

Figure 21. $\mathcal{N}$=2 scalar-photon tadpole: $\Sigma_{(\hat\phi_b)}(p)_{\rm tad}$.

Explicit computation based on these diagrams then gives the following two point functions ($\Sigma^{\dot\alpha\alpha}_{\rm NCSYM}$ and $\Sigma_{(\hat\phi)_{\rm NCSYM}}$) for noncommutative photino $\hat\lambda$ and adjoint scalar $\hat\phi$:

$$\begin{aligned}\Sigma^{\dot\alpha\alpha}_{\rm NCSYM} &= \Sigma^{\dot\alpha\alpha}_{\rm NCSYM_{bubble}}\\ &= \bar\sigma^\mu p_\mu \frac{1}{(4\pi)^2}\left((4\pi\mu^2)^{2-\frac{D}{2}}(p^2)^{\frac{D}{2}-2}(2-D)\Gamma\left(2-\frac{D}{2}\right)\mathrm{B}\left(\frac{D}{2}-1,\frac{D}{2}-1\right)\Big|_{D\to4-\epsilon}+4I^0_K\right),\end{aligned} \tag{F.19}$$

$$\Sigma_{(\hat\phi)_{\rm NCSYM}} = \Sigma_{(\hat\phi)_{\rm NCSYM_{bubble}}} + \Sigma_{(\hat\phi)_{\rm NCSYM_{tadpole}}}, \tag{F.20}$$

$$\begin{aligned}\Sigma_{(\hat\phi)_{\rm NCSYM_{bubble}}} &= p^2\frac{4}{(4\pi)^2}\\ &\cdot\left(-(4\pi\mu^2)^{2-\frac{D}{2}}(p^2)^{\frac{D}{2}-2}\Gamma\left(2-\frac{D}{2}\right)\mathrm{B}\left(\frac{D}{2}-1,\frac{D}{2}-1\right)\Big|_{D\to4-\epsilon}-T_0+2I^0_K\right),\end{aligned} \tag{F.21}$$

$$\Sigma_{(\hat\phi)_{\rm NCSYM_{tadpole}}} = p^2\frac{16}{(4\pi)^2}T_0. \tag{F.22}$$

References

[1] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP **09** (1999) 032 [hep-th/9908142] [INSPIRE].

[2] T. Mehen and M.B. Wise, *Generalized $*$-products, Wilson lines and the solution of the Seiberg-Witten equations*, JHEP **12** (2000) 008 [hep-th/0010204] [INSPIRE].

[3] B. Jurčo and P. Schupp, *Noncommutative Yang-Mills from equivalence of star products*, Eur. Phys. J. **C 14** (2000) 367 [hep-th/0001032] [INSPIRE].

[4] C.P. Martin and D. Sánchez-Ruiz, *The one loop UV divergent structure of* U(1) *Yang-Mills theory on noncommutative R^4*, Phys. Rev. Lett. **83** (1999) 476 [hep-th/9903077] [INSPIRE].

[5] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, JHEP **02** (2000) 020 [hep-th/9912072] [INSPIRE].

[6] M. Hayakawa, *Perturbative analysis on infrared aspects of noncommutative QED on R^4*, Phys. Lett. **B 478** (2000) 394 [hep-th/9912094] [INSPIRE].

[7] M. Hayakawa, *Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on R^4*, hep-th/9912167 [INSPIRE].

[8] P. Schupp and J. You, *UV/IR mixing in noncommutative QED defined by Seiberg-Witten map*, JHEP **08** (2008) 107 [arXiv:0807.4886] [INSPIRE].

[9] J. Trampetić and J. You, *θ-exact Seiberg-Witten maps at order e^3*, Phys. Rev. **D 91** (2015) 125027 [arXiv:1501.00276] [INSPIRE].

[10] R. Horvat, A. Ilakovac, J. Trampetić and J. You, *On UV/IR mixing in noncommutative gauge field theories*, JHEP **12** (2011) 081 [arXiv:1109.2485] [INSPIRE].

[11] R. Horvat, A. Ilakovac, J. Trampetić and J. You, *Self-energies on deformed spacetimes*, JHEP **11** (2013) 071 [arXiv:1306.1239] [INSPIRE].

[12] R. Horvat, J. Trampetić and J. You, *Photon self-interaction on deformed spacetime*, Phys. Rev. **D 92** (2015) 125006 [arXiv:1510.08691] [INSPIRE].

[13] A. Armoni and E. Lopez, *UV/IR mixing via closed strings and tachyonic instabilities*, Nucl. Phys. **B 632** (2002) 240 [hep-th/0110113] [INSPIRE].

[14] D. Zanon, *Noncommutative $N = 1$, $N = 2$ super* U(N) *Yang-Mills: UV/IR mixing and effective action results at one loop*, Phys. Lett. **B 502** (2001) 265 [hep-th/0012009] [INSPIRE].

[15] F.R. Ruiz, *Gauge fixing independence of IR divergences in noncommutative* U(1), *perturbative tachyonic instabilities and supersymmetry*, Phys. Lett. **B 502** (2001) 274 [hep-th/0012171] [INSPIRE].

[16] C.P. Martin and C. Tamarit, *The Seiberg-Witten map and supersymmetry*, JHEP **11** (2008) 087 [arXiv:0809.2684] [INSPIRE].

[17] C.P. Martin, J. Trampetić and J. You, *Super Yang-Mills and θ-exact Seiberg-Witten map: absence of quadratic noncommutative IR divergences*, JHEP **05** (2016) 169 [arXiv:1602.01333] [INSPIRE].

[18] B.S. DeWitt, *Quantum Theory of Gravity. 2. The Manifestly Covariant Theory*, Phys. Rev. **162** (1967) 1195 [INSPIRE].

JHEP09(2016)052

[19] R.E. Kallosh, *The Renormalization in Nonabelian Gauge Theories*, Nucl. Phys. **B 78** (1974) 293 [INSPIRE].

[20] B.S. DeWitt, *A gauge invariant effective action*, NSF-ITP-80-31 [INSPIRE].

[21] B.S. DeWitt, *Dynamical Theory Of Groups And Fields*, [INSPIRE].

[22] J. Gomis and T. Mehen, *Space-time noncommutative field theories and unitarity*, Nucl. Phys. **B 591** (2000) 265 [hep-th/0005129] [INSPIRE].

[23] C.P. Martin, J. Trampetić and J. You, *Equivalence of quantum field theories related by the θ-exact Seiberg-Witten map*, Phys. Rev. **D 94** (2016) 041703 [arXiv:1606.03312] [INSPIRE].

[24] S. Ichinose, *BRS symmetry on background field, Kallosh theorem and renormalization*, Nucl. Phys. **B 395** (1993) 433 [INSPIRE].

[25] C.P. Martin, *Computing the θ-exact Seiberg-Witten map for arbitrary gauge groups*, Phys. Rev. **D 86** (2012) 065010 [arXiv:1206.2814] [INSPIRE].

[26] C.P. Martin and D.G. Navarro, *The hybrid Seiberg-Witten map, its θ-exact expansion and the antifield formalism*, Phys. Rev. **D 92** (2015) 065026 [arXiv:1504.06168] [INSPIRE].

[27] A.F. Ferrari et al., *Superfield covariant analysis of the divergence structure of noncommutative supersymmetric QED$_4$*, Phys. Rev. **D 69** (2004) 025008 [hep-th/0309154] [INSPIRE].

[28] L. Álvarez-Gaumé and M.A. Vazquez-Mozo, *General properties of noncommutative field theories*, Nucl. Phys. **B 668** (2003) 293 [hep-th/0305093] [INSPIRE].

[29] A.F. Ferrari et al., *Towards a consistent noncommutative supersymmetric Yang-Mills theory: Superfield covariant analysis*, Phys. Rev. **D 70** (2004) 085012 [hep-th/0407040] [INSPIRE].

[30] I. Jack and D.R.T. Jones, *Ultraviolet finiteness in noncommutative supersymmetric theories*, New J. Phys. **3** (2001) 19 [hep-th/0109195] [INSPIRE].

[31] A. Santambrogio and D. Zanon, *One loop four point function in noncommutative $N = 4$ Yang-Mills theory*, JHEP **01** (2001) 024 [hep-th/0010275] [INSPIRE].

[32] M. Pernici, A. Santambrogio and D. Zanon, *The one loop effective action of noncommutative $N = 4$ super Yang-Mills is gauge invariant*, Phys. Lett. **B 504** (2001) 131 [hep-th/0011140] [INSPIRE].

[33] I.L. Buchbinder and I.B. Samsonov, *Noncommutative $N = 2$ supersymmetric theories in harmonic superspace*, Grav. Cosmol. **8** (2002) 17 [hep-th/0109130] [INSPIRE].

[34] M. Hanada and H. Shimada, *On the continuity of the commutative limit of the 4d $N = 4$ non-commutative super Yang-Mills theory*, Nucl. Phys. **B 892** (2015) 449 [arXiv:1410.4503] [INSPIRE].

[35] J.C. Collins, *Renormalization: an introduction to renormalization, the renormalization group, and the operator product expansion*, Cambridge monographs on mathematical physics, Cambridge University Press (1985), pg. 380, [ISBN: 0521311772].

[36] Wolfram Research Inc., *Mathematica*, Version 8.0, Champaign, IL, U.S.A. (2010)

[37] J. Martin-Garcia, *xAct*, http://www.xact.es/.

JHEP09(2016)052